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GRAY CODE FOR GENERATING TREE OF PERMUTATION WITH THREE CYCLES

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ABSTRACT

This paper present a new Gray code formed from generation tree of n-length permutation with three cycles. Gray code is a list of all the objects arranged such that, there are only minor differences between one object to another object. To be effective, listing of all objects is done with successive generation, where the structure-i^{th} is determined by structure-$(i-1)^{th}$. The listing algorithm is done by finding a method or algorithm for listing all objects in a certain order without any repetition and without losing any of the objects, so that two successive objects differ slightly. Combinatorial Gray code is widely used, for example, in circuit testing, hardware and software testing, encryption, data compression, and games. The purpose of this research is to develop a new Gray code that formulated from generating tree of n-length permutation with three cycles proposed by Puspitodjati. The Gray code formulated by traversing the generating tree of permutation with three cycles. It is a modification of Bernini's Gray code formulation for Catalan number generating tree. The Gray code then analyzed by measuring the Hamming distance of each two successive words of the list and it is of a Hamming 1.

Keywords: combinatorial gray code, generating tree, n-length permutation with three cycles, hamming distance.

INTRODUCTION

Combinatorics is the study of mathematical properties of discrete structures. Combinatorics has four main branches of science: enumeration, generation, listing, and optimization. Initially, combinatorics studies only enumeration of combinatorial objects in a combinatorial class. Combinatorial class or class is called the set of combinatorial objects, while the combinatorial objects itself called object.

Listing of a combinatorial object, is to build an algorithm to generate all possible structures of S where the i^{th}-structure is determined from the $(i-1)^{th}$ -structure or the structure of length n associated with the structure of length $(n-1)$, and have a small Hamming distance.

List all combinatorial objects arranged in a way that there is only a minor difference between one object and the next object, known as combinatorial Gray code. Savage in [1] defines 'slightly different' as different in 'some pre-specified, small way'. Ruskey in [2] defines 'slightly different' of one object to the next object in general as a member of the closeness relations of a class of objects being listed. One of a closeness relation is Hamming distance, where two successive objects differ in a certain constant. If the constant number is minimal then the Gray code is said to be optimal. A Gray code is said to be cyclic if the first object and the last objects differ slightly, or in general, it satisfies the closeness relation. Bernini in [3] defines a Gray code as a finite set of string lists, with infinite string length, so that the Hamming distance between two consecutive strings is limited. Bernini also discusses a family of combinatorial Gray code based on generating trees, which codes obtained by traversing nodes in the generating tree.

A previous research on Gray code of permutations with a given cycles, has performed by Baril in [4]. Baril build Gray code listing algorithm wherein the formulation is divided into five cases based on the relation between the number of cycles and the size of permutations. Mapping of the permutations with cycle to the array of transposition, produce a list of string where every consecutive two strings differs at most in two positions, and strictly defined by Baril as a Gray code.

Bernini in [3] and Vajnovszki in [5] developed a Gray code of several combinatorial objects, including permutations based on the succession rules (rules of branching tree) of an ECO (Enumerating Combinatorial Objects) system. Bernini in [3] defines a procedure to encode and to list objects of Catalan numbers such that two consecutive objects differ by one-digit. Gray structure is defined as an object of combinatorial class which construction meets a succession rules that have a stability property.

Gray code that developed from generating tree for n-length permutation with two cycles has been proposed by Puspitodjati as in [6].

Gray code resulting from the transformation, as Baril in [4], as well as the encoding of the succession rules (generating trees) by Bernini in [3], became the basic idea of this paper. This paper builds a Gray code listing formulation with the encoding of generating tree of permutation with three cycles as a combinatorial object. The generating tree that becomes combinatorial object of this research is Puspitodjati's generating tree of permutations with cycle as in [7].

Puspitodjati in [6] also developed a Gray code that developed from generating tree for n-length permutation with two cycles. But the way Puspitodjati formulated the Gray code of generating tree of n–length permutation with two cyles, couldn't be applied for generating tree n-length permutation with three cyles.

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While Bernini's approach in [3], performed a traversal with a particular procedure on such tree and found regularity on nodes listing at a certain level. Bernini shows that interestingly, the regularity is inherited by the regularity of nodes at the previous level. This approach then becomes an idea to formulate a new Gray code that present in this paper.

GENERATING TREE

A generating tree is an infinite rooted tree. The root (at level 0) labelled as e_0. Each branch has a node labelled (*k*) and has a relation with the next node at a greater level. The systematization of the relation of each node to the next node summarized in a rule called the succession rule Banderier in [8]. A generating tree for a class of combinatorial objects could apply if there is a bijection from the size of the object to the number of nodes at level *d* of the tree. As an example is the generating tree for Catalan numbers, as in Figure-1.

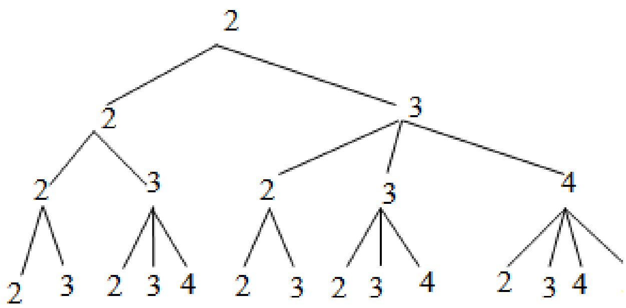

Figure-1. Catalan number generating tree.

The tree has systematized and the succession follows the rules as in (1) as follows:

$$\Omega = \begin{cases} (2) \\ (k) \rightarrow (2)(3)\ldots(k)(k+1), \quad k \geq 2 \end{cases} \tag{1}$$

Bernini in [3] proposed a formulation of visiting nodes on the generating tree of objects count by Catalan number as in Figure-1, thus formed a Gray code, with the introduction of a shifted production function *s(k, i)* as follows:

$$\begin{cases} s(k,2) = < 2, k+1, k, k-1, \ldots, 4, 3 > \\ s(k,i) = < i, i+1, \ldots, k-1, k, k+1, 2, 3, \ldots, i-1 > \end{cases} \tag{2}$$

The four-digit Gray codes generated from (2) for the generating tree of the Catalan numbers are: [2222, 2223, 2233, 2234, 2232, 2332, 2334, 2333, 2343, 2344, 2345, 2342, 2322, 2323].

PERMUTATION WITH CYCLES

A permutation of a set [*n*] = {1, 2, 3,..., *n*} is a one-one onto function π: [*n*]→ [*n*]. Permutation can be written in a one line notation, which sequence shows the sequence of mapping of [*n*]. For example, the 6-legth permutation 421365 means $\pi(1)$ mapped to 4, $\pi(2)$ mapped to 2, through $\pi(6)$ mapped to 5.

A cycle of length *m* in a permutation is a sequence of distinct elements $a_1, a_2, \ldots, a_m$ such that $a_i = \pi(a_{i-1})$ for *i* = *2, 3,..m* and $a_1 = \pi(a_m)$. Such cycle is written as $(a_1, a_2, \ldots, a_m)$. All permutation can be decomposed into the disjoint unions of their cycles. For example, a one line notation of permutation 421365 would be in a cycle notation as (1 4 3) (2) (5 6).

GRAY CODE LISTING FOR THE GENERATING TREE OF PERMUTATION WITH THREE CYCLES

Puspitodjati in [7] formulated the generation of *n*-length permutations with *m* cycles. Based on the formulation, generating trees of *n*-length permutation with three cycles for the first three levels is presented in Figure-2. The generating tree in Figure-2 can be described according to the ECO succession rule (3) from [7].

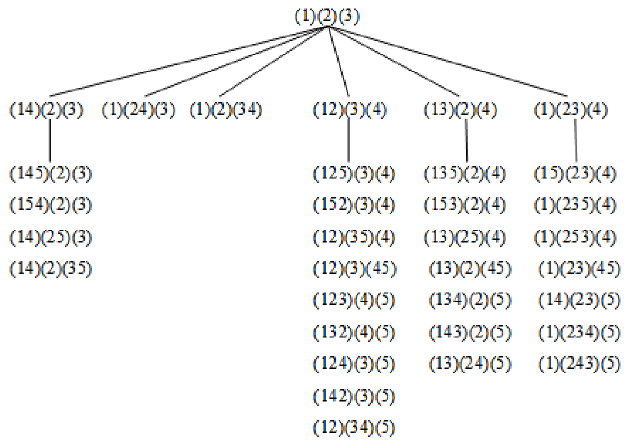

Figure-2. S_{n5} generating tree

When the tree in [7] coded such that nodes labeled as the number of their children, the generating tree of *n*-length permutations with three cycles then has a succession rules formulated in (3) as follows:

$$\Omega_3 = \begin{cases} (0) \rightarrow (1)(2)(3) \\ (k) \rightarrow (k+1)(k+2)\ldots(level+2) \end{cases} \tag{3}$$

The new tree formed from the generating tree of *n*-length permutation with three cycles, to level 5, can be seen in Figure-3.

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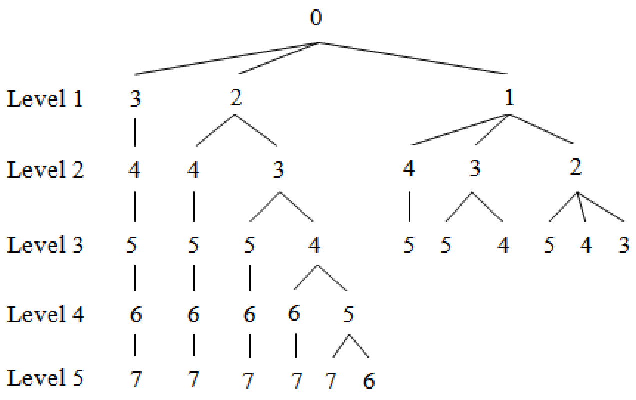

Figure-3. The summary of $S_{n,3}$ generating tree.

Using a similar Bernini strategy as in [3], and conducting some modifications, this research formulates the rules of Gray code generation for the generating tree of permutation with three cycles as will be described in the following section.

GRAY CODE LISTING ALGORITHM FOR THE GENERATING TREE OF $S_{n,3}$ PERMUTATION

The new combinatorial Gray code of Puspitodjati's generating tree of *n*-legth permutation with three cycles in [8] formulated using these notations and definitions:

L_d = list of code with length *d* digit/level *d*
$L_d = [l_{d,1}, l_{d,2}, l_{d,3}, \ldots]$
M = | L_{d-1} |= cardinality of L_{d-1}
$\bar{x}$ = most right digit *x*
O = multiple concatenation
$\circ$ = concatenation

If *L* is a list of code, then: *first*(*L*) states the first element of *L* and *last*(*L*) states the last element of *L*

Theorem-1: List L_d formed in the following manner is a Gray code with Hamming distance of 1

$$L_1 = [3, 2, 1] \quad L_d = \mathop{\mathrm{O}}_{i=1}^{M} L_{d,i} \text{ , } d > 1\text{, where } M = |L_{d-1}| \tag{4}$$

$max(d) = [3 + d - 1]$ and $L_{d,i}$ defined by:

$$L_{d,1} = l_{d-1,1} \circ s\left(\overline{l_{d-1,1}}, \max(d)\right) \tag{4}$$

$$L_{d,i} = l_{d-1,i} \circ s\left(\overline{l_{d-1,i}}, last\left(\overline{L_{d,i-1}}\right)\right) \tag{5}$$

$$\text{where: } s(k,j) = \begin{cases} (j, j-1, j-2, \ldots, k+1) & if \quad j = \max(d) \\ (j, \max(d), \max(d)-1, \ldots, k+1) & if \quad j < \max(d) \end{cases}$$

Proof

Mathematical proof is done by induction on *d*. For base *d* = 1, then L_1 = [3, 2, 1], it is clearly different by 1 digit or have a Hamming distance of 1. For inductive hypothesis: assume that it is true for *d* = 2, ..., *r*, then formula (4) and (5) applies.

Inductive step

for *d* > *r*+1, each element $L_{d,i}$ is formed with one digit different, so it only needs to proof that $last(L_{d,i})$ and $first(L_{d,i+1})$, for $1 \le i \le M-1$, are different by 1 digit. If *J* is the last element of $s(\overline{l_{d-1,i}} last(\overline{L_{d,i-1}}))$, then $last(L_{d,i}) = l_{d-1,i} \circ J$. Whereas $L_{d,i+1}$ is obtained through

$$L_{d,i+1} = l_{d-1,i+1} \circ s(\overline{l_{d-1,i}}, last(\overline{L_{d,i+1}}))$$
$$= l_{d-1,i+1} \circ s\left(\left(\overline{l_{d-1,i+1}}\right), J\right)$$

By the definition of *shifted list of the successors* $s(k, j)$ the element of *first* $s(k,j) = j$, becomes $first(L_{d,i+1}) = l_{d-1,i+1} \circ J$.

So $l_{d-1,i}$ and $l_{d-1,i+1}$ are different by 1 digit, by mathema-tical induction, as well as $last(L_{d,i})$ and $first(L_{d,i+1})$ are different by 1 digit.

GRAY CODE LISTING FOR THE GENERATING TREE OF $S_{n,3}$ PERMUTATION

Based on the algorithm described above, the Gray code for a single digit (*d* = 1) has been determined as L_1= [3, 2, 1]. Furthermore, the Gray code listing is formed using an algorithm that has been formulated with one digit Gray code as input to obtain a two-digits Gray code (*d*=2). The *d*-digits Gray code listing is then obtained by (*d-1*) digits Gray code input.

Here are the steps of Gray code listing for a generating tree of *n*-length permutation with three cycles:

Gray code for a generating tree of *n*-length permutation with three cycles:

One digit Gray code

d= 1: $L_1 = [3, 2, 1]$

Two digits Gray code

d = 2, then max (*d*) = 3 + 2 – 1 = 4. The two digits Gray code L_2 = [$L_{2,1}$, $L_{2,2}$, $L_{2,3}$], with a set of one digit Gray code $L_1 = [3, 2, 1]$ as input, resulted as follows.

$L_{2,1}$ has $l_{1,1} = 3$, $\overline{l_{1,1}} = 3$, and from (4)

$$L_{2,1} = l_{1,1} \circ s\left(\overline{l_{1,1}}, \max(d)\right) = 3 \circ s(3,4) = 3 \circ \langle 4 \rangle = 34$$

Furthermore by (5)

$L_{2,2}$ could be determined where

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$last\left(\overline{L_{2,1}}\right)=\overline{34}=4,\;\; l_{1,2}=2$ and $\overline{l_{1,2}}=2$ are obtained from $L_1=[3,2,1]$, so that

$L_{2,2}=l_{1,2}\circ s\left(\overline{l_{1,2}},last\left(\overline{L_{2,1}}\right)\right)=2\circ \mathrm{s}(2,4)=24,23$.

Likewise, the member of $L_{2,3}$ where

$last\left(\overline{L_{2,2}}\right)=\overline{23}=3$, from $L_1=[3,2,1]$ we obtain

$l_{1,3}=1$, and $\overline{l_{1,3}}=1$,

$L_{2,3}=l_{1,3}\circ s\left(\overline{l_{1,3}},last\left(\overline{L_{2,2}}\right)\right)=1\circ \mathrm{s}(1,3)=13,14,12,$ so that $L_2=[34,24,23,13,14,12]$

Three digits Gray code

The three digits Gray code is formed by utilizing the set of two digits Gray code $L_2=[34,24,23,13,14,12]$

The determination of $L_{3,1}$ is obtained with regard to $l_{2,1}=34$, $\overline{l_{2,1}}=4$ and max(d)=5, then

$L_{3,1}=l_{2,1}\circ s\left(\overline{l_{2,1}},\max(d)\right)=34\circ \mathrm{s}(4,5)=34\circ <5>=345$

Furthermore by (5), since $last\left(\overline{L_{3,1}}\right)=\overline{345}=5$, and

$l_{2,2}=24$, $\overline{l_{2,2}}=4$, then

$L_{3,2}=l_{2,2}\circ s(\overline{l_{2,2}},last(\overline{L_{3,1}}))=24\circ s(4,5)=245$.

As well as the member of $L_{3,3}$ where

$last\left(\overline{L_{3,2}}\right)=\overline{245}=5,$ and $l_{2,3}=23$, $\overline{l_{2,3}}=3,$ so that

$L_{3,3}=l_{2,3}\circ s\left(\overline{l_{2,3}},last\left(\overline{L_{3,2}}\right)\right)=23\circ \mathrm{s}(3,5)=235,234$

The determination of L_{34} regarding the value of $l_{2,4}=13$, $\overline{l_{2,4}}=3,\;\; last\left(\overline{L_{3,3}}\right)=\overline{234}=4$ gives

$L_{3,4}=l_{2,4}\circ s\left(\overline{l_{2,4}},last\left(\overline{L_{3,3}}\right)\right)=13\circ \mathrm{s}(3,4)=134,135$

The determination of L_{35} regarding the value of $l_{2,5}=13$, $\overline{l_{2,5}}=3$ and $last\left(\overline{L_{3,4}}\right)=\overline{135}=5$ gives

$L_{3,5}=l_{2,5}\circ s\left(\overline{l_{2,5}},last\left(\overline{L_{3,4}}\right)\right)=14\circ \mathrm{s}(4,5)=145$

The determination of $L_{3,6}$ regarding the value of $l_{2,6}=13$, $\overline{l_{2,6}}=3$, and $last\left(\overline{L_{3,5}}\right)=\overline{145}=5$, gives

$L_{3,6}=l_{2,6}\circ s\left(\overline{l_{2,6}},last\left(\overline{L_{3,5}}\right)\right)=125,124,123$

Hence

L_3 =[345, 245, 235, 234, 134, 135, 145, 125, 124, 123]

If the calculation procedure continues then the Gray code for the generating tree of *n* permutation with three cycles for 4 digits is as follows:

L_4= [3456, 2456, 2356, 2346, 2345, 1345, 1346, 1356, 1456, 1256, 1246, 1245, 1235, 1234]

The algorithm corresponds to the formulation of Gray code for the generating tree of permutation with three cycles is as follows:

Algorithm-1

GraycodeGTS$_{n,3}$(d)
1. var *i*
2. begin
3. *L*=[3, 2, 1]
4. max(*d*)= 3+*d*-1
5. N = |*L*|
6. *L*(1)= *GraycodeGTS$_{n,3}$(d-1)* $\circ$ s(*L*(1), max(*d*))
7. for *i*= 2,…, *N*
8. *L*(*i*)= *GraycodeGTS$_{n,3}$(d)* $\circ$ s(*L*(*i*), *last*(*L*(*i*-1)))
9. end *i*
10. end

CONCLUSIONS

The listing Gray code for the generating tree of permutation n with three cycles is successfully formulated based on Puspitodjati's generating tree for permutation with 3 cycles. The formulation is a modification of the Gray code listing formulation proposed by Bernini [3]. The modifications is on the definition of shift production s(*k, j*) and the determination of max (*d*), resulting in the listing algorithm of Gray code for the generating tree of permutation n with three cycles.

This research has produced a complete generation algorithm; all objects appear and without repetition, and listed according to the concept of optimal Gray code, i.e. between two consecutive objects have a Hamming distance of 1.

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