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ANALYTICAL APPROXIMATE SOLUTIONS FOR THE HELMHOLTZ-DUFFING OSCILLATOR

Md. Alal Hosen[1,2] and M. S. H. Chowdhury[1]
[1]Department of Science in Engineering, Faculty of Engineering, International Islamic University Malaysia, Jalan Gombak, Kuala Lumpur, Malaysia
[2]Department of Mathematics, Rajshahi University of Engineering and Technology, Rajshahi, Bangladesh
E-Mail: sazzadbd@iium.edu.my

ABSTRACT

In the present paper, a new analytical technique is introduced for obtaining approximate periodic solutions of Helmholtz-Duffing oscillator. Modified Harmonic Balance Method (MHBM) is adopted as the solution method. A classical harmonic balance method does not apply directly for solving Helmholtz-Duffing oscillator. Generally, a set of difficult nonlinear algebraic equations is found when MHBM is applied. Investigating analytically for such kinds of nonlinear algebraic equations is a tremendously difficult task and cumbersome especially for large oscillation. In this study, the offered technique eradicates this aforementioned limitation and avoids numerical complexity. Using iterative homotopy perturbation method, only two or three iteration produces desired results even for large oscillation. It is remarkably important that a second-order approximate solution gives excellent agreement compared to exact ones.

Keywords: helmholtz-duffing oscillator, harmonic balance method, iterative homotopy perturbation method.

INTRODUCTION

The nonlinear Helmholtz oscillator has received considerable attention especially in the last decade in nonlinear sciences and engineering. The interest arises from large number of applications in the mathematical interpretation of the engineering problems such as ship dynamics, oscillation of the human eardrum, dynamics of a particle moving in a cubic potential and oscillations of one dimensional structural system with an initial curvature. Along with the rapid progress of nonlinear sciences, an intensifying interest among with scientists and researchers has been already proposed varieties of approximate and numerical solution methods to solve Helmholtz-Duffing oscillator. The harmonic balance method was employed by Thywle [1] to analyze the forced oscillation of a Helmholtz-Duffing oscillator. Based on the Jacobi elliptic function, an analytical solution was obtained by Cveticanin for a hardening and softening Helmholtz oscillator [2]. The symmetry breaking phenomenon for a general forced Helmholtz-Duffing oscillator was studied by Cao *et al.* [3]. More recently, the Homotopy perturbation method was employed by Leung and Guo [4-5] to obtain approximate solutions of the Helmholtz-Duffing oscillator. A large variety of variational and perturbative methods commonly used for nonlinear oscillatory systems especially for strongly nonlinear oscillators have been recently extended mostly by He for instance, one can refer to the Modified He's Homotopy Perturbation Method [6-8], He's Modified Lindsted-Poincare Method [9], He's Max-Min Approach Method [10], He's Energy Balance Method [11-14], He's Frequency Amplitude Formulation Method [15-16] and other classical perturbative and non-perturbative techniques including Homotopy Perturbation Method [17], Residue Harmonic Balance Method [18], Algebraic Method [19], Rational Energy Balance Method [20], Iteration Method [21-24], Harmonic Balance Method [25-31], Rational Harmonic Balance Method [32] and so on. However, most of these methods have been considered only first-order approximation solution which leads to low accuracy. In addition, the aforementioned methods also do not have this ability to gain the solution in high precision. Furthermore, the solution procedures are tremendously difficult task and cumbersome especially for obtaining higher order approximation.

In this situation, the approximate periodic solutions for the Helmholtz-Duffing oscillator are studied employing MHBM. Generally, a set of difficult nonlinear algebraic equations are found when MHBM is formulated. Sometime analytical solutions of these algebraic equations fail especially for large amplitude. In article Alam *et al.* [30], such nonlinear algebraic equations are solved in powers of a small parameter. The solutions derived (Alam *et al.* [30]) for *Duffing* equation agree with numerical solutions when $[x(0)=a_0, \dot{x}(0)=0]$, $a_0=\mathrm{O}(1)$. Sometimes, higher approximations also fail to measure the desired results when $a_0 \gg 1$. In present study, this limitation is removed. The second-order analytical approximate periodic solution has been obtained for nonlinear Helmholtz-Duffing oscillator. Considering the interesting property that the proposed technique not only provides accurate results but also it is more convenient and effective for solving more complex nonlinear problems.

The asymmetric nonlinear Helmholtz-Duffing oscillator is separated into two auxiliary equations applicable in positive and negative directions. Analytical expressions are then shown for the natural frequency of the oscillation. Dynamic responses are compared in time domain and also accuracy of the approximate solutions is evaluated. Error analysis is then carried out and performances of the different solution techniques are compared.

SOLUTION PROCEDURE

A nonlinear second-order differential equation is of the form

$\ddot{x}+\omega_0^2 x=-\varepsilon f(x)$ and the initial condition

$$[x(0)=a_0,\dot{x}(0)=0], \tag{1}$$

where $f(x)$ is a nonlinear function such that $f(-x)=-f(x)$, $\omega_0\geq 0$ and ε is a constant.

Consider a second-order periodic solution of Equation (1) is in the form

$$x=a_0(\rho\cos(\omega t)+u\cos(3\omega t)), \tag{2}$$

where a_0, ρ and ω are constants. If $\rho=1-u$ and the initial phase $\varphi_0=0$, solution Equation (2) readily satisfies the initial conditions $[x(0)=a_0,\dot{x}(0)=0]$.

Substituting Equation (2) into Equation (1) and expanding $f(x)$ in a Fourier series, it converts to an algebraic identity

$$\begin{aligned}&a_0[\rho(\omega_0^2-\omega^2)\cos(\omega t)+u(\omega_0^2-9\omega^2)\cos(3\omega t)]\\&=-\varepsilon[F_1(a_0,u,\cdots)\cos(\omega t)+F_3(a_0,u,\cdots)\cos(3\omega t)]\end{aligned} \tag{3}$$

Now equating the like coefficients of equal harmonic terms of Equation (3), the following nonlinear algebraic equations are found

$$\rho(\omega_0^2-\omega^2)=-\varepsilon F_1,\quad u(\omega_0^2-9\omega^2)=-\varepsilon F_3 \tag{4}$$

Applying the first equation of Equation (4), ω^2 is eliminated from other equation of Equation (4). Substitute $\rho=1-u$, and simplification, second equations of Equation (4) reduce to the following form

$$u=G_1(\omega_0,\varepsilon,a_0,u,\cdots,\lambda_0), \tag{5}$$

where G_1 exclude respectively the linear terms of u.

Now applying the iterative homotopy perturbation method (**See Appendix**) to obtained the value of u is

$$u=u_0+u_1+u_2+u_3+\cdots \tag{6}$$

At long last, substituting the estimations of u from Equation (6) into the first equation of Equation (4), the frequency ω is determined. This completes the determination of all related unknowns for the proposed periodic solution as given in Equation (2).

EXAMPLE

The Helmholtz-Duffing oscillator

The Helmholtz-Duffing oscillator is considered in this section with the governing equation given by

$$\ddot{x}+x+(1-\lambda)x^2+\lambda x^3=0,\ x(0)=a_0,\ \dot{x}(0)=0. \tag{7}$$

Since the behavior of an asymmetric nonlinear oscillator is different in positive and negative directions, the equation can be conveniently studied in two parts [4-5]

$$\ddot{x}+x+(1-\lambda)x^2 sg(x)+\lambda x^3=0,\ \text{for } x\geq 0, \tag{8}$$

$$\ddot{x}+x-(1-\lambda)x^2 sg(x)+\lambda x^3=0,\ \text{for } x\leq 0, \tag{9}$$

λ is an asymmetric parameter representing the extend of asymmetry. For $\lambda=0$ the equation governs motion of a Helmholtz oscillator and for $\lambda=1$ it denotes a Duffing differential equation. The system is assumed to oscillate between an asymmetric limit zone $[-b,a]$, for positive and b. Both $x=a$ and $x=-b$ represent the turning points in which $\dot{x}=0$, a and b are an unknown amplitude to be determined.

We observed that, the application of classical harmonic balance method directly in Equation (8) does not work. For applicable of classical harmonic balance method, the quadratic term of Equation (8) *i.e.* $(1-\lambda)x^2$ can be expanded in a Fourier series as

$$\begin{aligned}(1-\lambda)x^2=\sum_{n=0}^{\infty}b_{2n+1}(1-\lambda)x^2=b_1\cos(\omega t)\\+b_3\cos(3\omega t)+\cdots\end{aligned} \tag{10}$$

Herein b_1, b_3, $\cdots$ are calculated by the integration

$$b_{2n+1}=\frac{4}{\pi}\int_0^{\pi/2}(1-\lambda)x^2\cos[(2n+1)\varphi]d\varphi, \tag{11}$$

setting $\varphi=\omega t$. Now let us consider the first-order approximate solution of Equation (8) is

$$x_a=a_0\cos(\omega_a t) \tag{12}$$

From Equation (11) and Equation (12) we obtained b_1, $b_3,\cdots$,

$$b_1=\frac{8a_0^2(\lambda-1)}{3\pi}, \tag{13}$$

$$b_3=\frac{8a_0^2(\lambda-1)}{15\pi}, \tag{14}$$

and so on.

Now applying Equation (12) and Equation (10) along with Eqs. (13)-(14) into the Equation (8) and equating the coefficient of $\cos(\omega_a t)$ the accompanying mathematical expression is acquired

$$1+\frac{3a_0^2\lambda}{4}-\omega_a^2+\frac{8a_0(\lambda-1)}{3\pi}=0 \tag{15}$$

After simplification of Equation (15), the approximate natural frequency is the following

$$\omega_a=\sqrt{1+\frac{3a_0^2\lambda}{4}+\frac{8a_0(\lambda-1)}{3\pi}} \tag{16}$$

Without repeating the solution process, we can obtain for negative direction for the trial function $x_b = b_0 \cos(\omega_b t)$ as

$$\omega_b = \sqrt{1 + \frac{3b_0^2\lambda}{4} + \frac{8b_0(1-\lambda)}{3\pi}} \tag{17}$$

Also let us consider the second-order approximate solution of Equation (8) is

$$x_a = a_0 \cos(\omega_a t) + a_0 u(\cos(3\omega_a t) - \cos(\omega_a t)) \tag{18}$$

From Equation (18) and Equation (10) we obtained b_1, $b_3, \cdots$,

$$b_1 = \frac{8a_0^2(1-\lambda)(35-56u+48u^2)}{105\pi}, \tag{19}$$

$$b_3 = \frac{8a_0^2(1-\lambda)(21+120u-176u^2)}{315\pi}, \tag{20}$$

and so on.
Now substitute Equation (18) and Eqs. (19)-(20) into the Equation (8) and then equating the coefficients of $\cos(\omega_a t)$ and $\cos(3\omega_a t)$ to zero are the following

$$1-u+\left(1-2u+3u^2-2u^3\right)\frac{3a_0^2\lambda}{4} - (1-u)\omega_a^2 + \frac{8a_0(1-\lambda)(35-56u+48u^2)}{105\pi} = 0 \tag{21}$$

$$u+\left(1+3u-9u^2+8u^3\right)\frac{a_0^2\lambda}{4} - 9u\omega_a^2 + \frac{8a_0(1-\lambda)(21+120u-176u^2)}{315\pi} = 0 \tag{22}$$

After disentanglement, Equation (21) diminish to

$$\omega_a^2 = 1+\left(1-2u+3u^2-2u^3\right)\frac{3a_0^2\lambda}{4(1-u)} + \frac{8a_0(1-\lambda)(35-56u+48u^2)}{105\pi(1-u)} \tag{23}$$

By omitting of ω_a^2 from Equation (22) by using Equation (23) and some modification, the following nonlinear algebraic equation of u is

$$\begin{aligned} f(u): & \frac{8a_0}{15\pi}(\lambda-1) - \frac{a_0^2\lambda}{4} + \left(8 + \frac{25a_0^2\lambda}{4} - \frac{752a_0}{35\pi}(\lambda-1)\right)u \\ & - \left(8 + \frac{9728a_0}{315\pi}(1-\lambda) + \frac{21a_0^2\lambda}{2}\right)u^2 \\ & + \left(16a_0^2\lambda + \frac{256a_0}{9\pi}(1-\lambda)\right)u^3 - \frac{23a_0^2u^4\lambda}{2} = 0 \end{aligned} \tag{24}$$

Now applying the iterative homotopy perturbation method (**See Appendix**) to obtained the value of u from Equation (24) is

$$u = u_0 + u_1 + u_2 + u_3 + \cdots, \tag{25}$$

where u_0 is an initial approximation and the unknowns u_1, u_2, $u_3 \cdots$ are

$$u_1 = -\frac{f(u_0)}{f'(u_0)}$$

$$u_2 = -\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2$$

$$u_3 = \frac{1}{f'(u_0)}\left(\frac{1}{6}\left(\frac{f(u_0)}{f'(u_0)}\right)^3\right)f'''(u_0) + \frac{f(u_0)}{f'(u_0)}\left(-\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2\right)$$

and so on.
Now putting the value of u from Equation (25) into Equation (23), the approximate natural frequency is determined the following

$$\omega_a = \sqrt{1+\left(1-2u+3u^2-2u^3\right)\frac{3a_0^2\lambda}{4(1-u)} + \frac{8a_0(1-\lambda)(35-56u+48u^2)}{105\pi(1-u)}} \tag{26}$$

Without repeating the solution process, we can obtain for negative direction for the trial function $x_b = b_0 \cos(\omega_b t) + a_0 u(\cos(3\omega_b t) - \cos(\omega_b t))$ as

$$\begin{aligned} f(u): & \frac{8b_0}{15\pi}(1-\lambda) - \frac{b_0^2\lambda}{4} + \left(8 + \frac{25b_0^2\lambda}{4} - \frac{752b_0}{35\pi}(1-\lambda)\right)u \\ & - \left(8 + \frac{9728b_0}{315\pi}(\lambda-1) + \frac{21b_0^2\lambda}{2}\right)u^2 \\ & + \left(16b_0^2\lambda + \frac{256b_0}{9\pi}(\lambda-1)\right)u^3 - \frac{23b_0^2u^4\lambda}{2} = 0 \end{aligned} \tag{27}$$

Now applying the same iterative homotopy perturbation method (**See Appendix**) to obtain the value of u from Equation (27) and substitute into Equation (23) the approximate natural frequency for negative direction as the following.

$$\omega_b = \sqrt{1+\left(1-2u+3u^2-2u^3\right)\frac{3b_0^2\lambda}{4(1-u)}+\frac{8b_0(\lambda-1)(35-56u+48u^2)}{105\pi(1-u)}} \quad (28)$$

Table-1. Comparison the obtained frequencies between the previously existing frequencies and the exact frequencies for $\lambda = 0.9$.

a_0	ω_{ex}	ω_{MHBM}^{1st} $Er(\%)$	ω_{MHBM}^{2nd} $Er(\%)$	$\omega_{[11]}^{EBM}$ $Er(\%)$	$\omega_{[11]}^{EBM(Galerkin-Petrov)}$ $Er(\%)$	$\omega_{[4,5]}^{HFAF\ and\ HPM}$ $Er(\%)$
0.01	1.00003336	1.00003365 2.8999e-005	1.00003365 2.8999e-005	1.00003335 1.0575e-006	1.00003113 2.2300e-004	1.00003335 6.0755e-007
0.1	1.00334964	1.00336040 1.0724e-003	1.00335895 9.2789e-004	1.00335001 3.5992e-005	1.00312738 0.022152	1.00335106 1.4068e-004
0.5	1.08209182	1.08090914 0.109295	1.08044075 0.152581	1.08256404 0.043640	1.07725564 0.446928	1.08258341 0.045430
1.0	1.29782241	1.29380212 0.309771	1.28954738 0.637608	1.30241057 0.353527	1.28444299 1.030913	1.30244785 0.356400
5.0	4.17361022	4.22758564 1.293254	4.15285021 0.497411	4.25577625 1.968703	4.11954570 1.295389	4.25579387 1.969126
10	8.13109020	8.27631381 1.786028	8.11709681 0.172097	8.30612634 2.152677	8.02862044 1.260221	8.30613546 2.152789
50	40.22955916	41.09132917 2.142131	40.27851570 0.121692	41.12172027 2.217675	39.71819248 1.246264	41.12172212 2.217680
100	80.40995076	82.16445296 2.181946	80.53776157 0.158948	82.19486707 2.219770	79.40820415 1.245799	82.19486800 2.219771

Note: In Table 1, ω_{MHBM}^{1st} and ω_{MHBM}^{2nd} represent first- and second-order approximate natural frequencies obtained by MHBM. ω_{ex} represents exact frequency which is stated in Askari H. et al. [11]. $\omega_{[11]}^{EBM}$, $\omega_{[11]}^{EBM(Galerkin-Petrov)}$ and $\omega_{[4,5]}^{HFAF\ and\ HPM}$ indicate the approximate frequencies obtained previously in Askari H. et al. [11] and Leung A.Y.T. et al. [4, 5]. $Er(\%)$ denotes percentage error which has calculated by the relation $\left|\frac{\omega_i(a_0)-\omega_e(a_0)}{\omega_e(a_0)}\right| \times 100$ where $i = 1, 2, 3$.

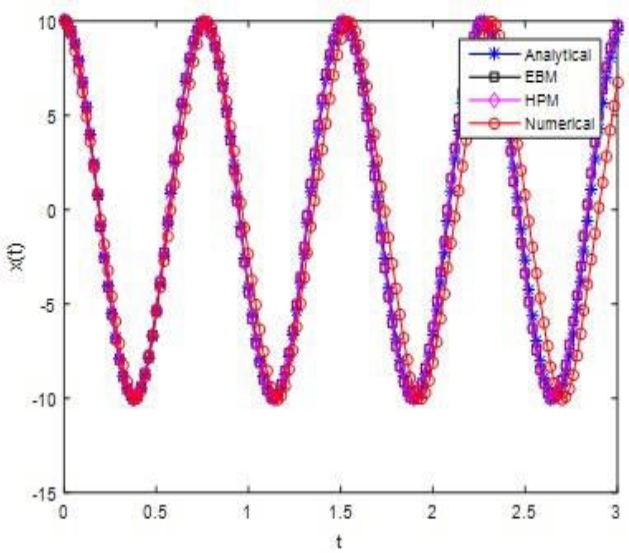

Figure-1. A Comparison between the first-order approximate solutions of Equation (7) for $\lambda = 0.9$ and $a_0 = 10$ together with corresponding previously existing solutions and numerical solutions.

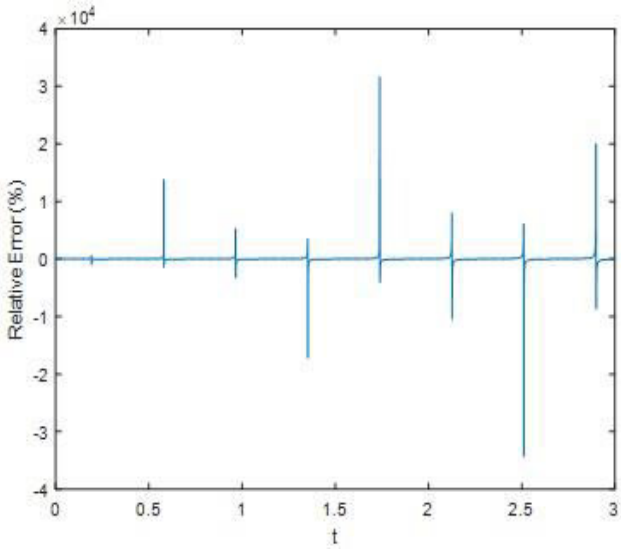

Figure-2. Relative error (%) for first-order approximate solutions of Equation (7) for $\lambda = 0.9$ and $a_0 = 10$.

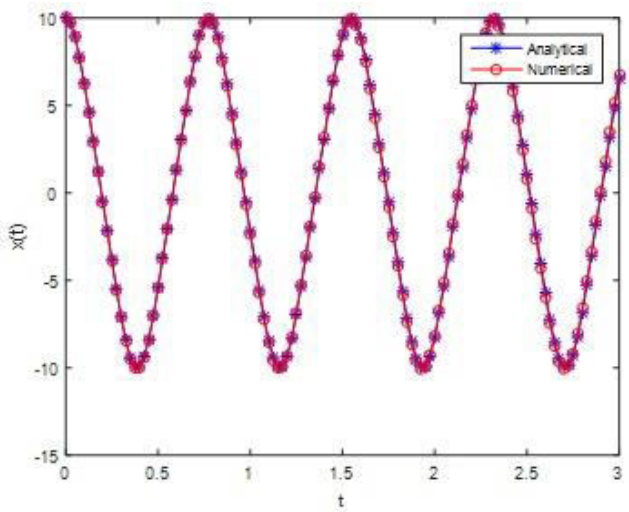

Figure-3. A Comparison between the second-order approximate solutions of Equation (7) for $\lambda = 0.9$ and $a_0 = 10$ together with corresponding numerical solutions.

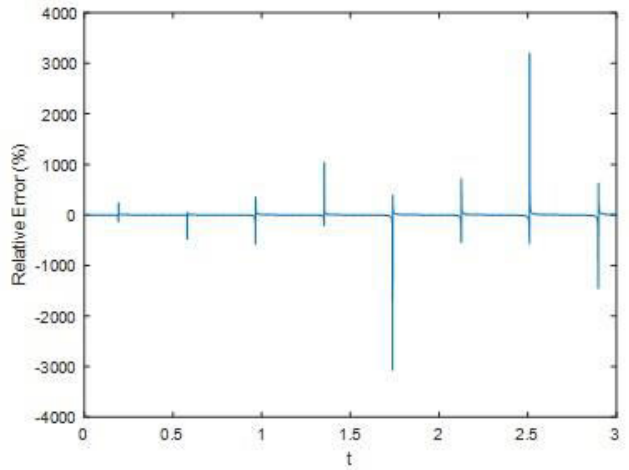

Figure-4. Relative error (%) for second-order approximate solutions of Equation (7) for $\lambda = 0.9$ and $a_0 = 10$.

RESULTS AND DISCUSSIONS

The approximation solutions and their relative errors (%) have been obtained by applying MHBM for the Helmholtz-Duffing oscillator. Comparing all approximation results with previously existing results and their corresponding numerical values are shown in Table-1 and Figures 1-4. It can clearly be seen that the accuracy of the results obtained by offered technique (second-order approximation) is almost similar to exact solutions and better than those obtained previously by several authors. Moreover, it is mentioned in [4, 5, 11] that the relative error (%) of approximate natural frequencies have been increased with increasing initial amplitude. In addition, the solution procedures of many existing methods are tremendously difficult task and cumbersome especially for obtaining the higher approximations. On the other hand, the offered method gives excellent agreement even for large amplitude. It is noted that the solution procedure of the proposed method is simple, straightforward, quite easy and highly efficient. The advantages of this method include its analytical simplicity and computational efficiency, and the ability to objectively find better results for many other oscillatory problems arising in nonlinear sciences and engineering.

CONCLUSIONS

In this paper, an efficient analytical technique has been introduced based on a MHBM to determine approximate periodic solutions of the Helmholtz-Duffing oscillator. In comparison with corresponding numerical results, we have seen that the approximate solutions show a good agreement with exact solutions. Moreover, in compared with previously published methods the determination procedure of approximate solutions is straightforward and simple. The high accuracy and validity of approximate solutions assured as about the results and reveal this method can be used easily for strongly nonlinear oscillators. To entirety up, we can say that the technique introduced in this study for solving strongly nonlinear Helmholtz-Duffing oscillator can be considered as powerful, an efficient alternative of the previously existing methods.

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APPENDIX

A higher-order nonlinear algebraic equation is of the form

$$f(x) = 0 \tag{A1}$$

Consider the nonlinear algebraic equation Equation (A1), and we construct a homotopy $H : R \times [0,1] \to R$ which satisfy

$$\begin{aligned} &H(x, p) = f(x) - f(x_0) + pf(x_0) = 0, \\ &x \in R, \quad p \in [0,1] \end{aligned} \tag{A2}$$

where p is embedding parameter, x_0 is an initial approximation of Equation (A1). Hence, it is obvious that

$$H(x,0) = f(x) - f(x_0) = 0 \tag{A3}$$

$$H(x,1) = f(x) = 0 \tag{A4}$$

and the changing process of p from 0 to 1, refers to $H(x, p)$ from $H(x,0)$ to $H(x,1)$. Applying the perturbation technique Javidi, [24], due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of Equation (A2) can be express as a series in p

$$x = x_0 + x_1 p + x_2 p^2 + x_3 p^3 + \cdots \tag{A5}$$

When $p \to 1$, Equation (A2) corresponds to Equation (A1) and Equation (A5) becomes the approximate solution of equation (A1), that is [24].

$$x = \lim_{p \to 1} x = x_0 + x_1 + x_2 + x_3 + \cdots \tag{A6}$$

and in [24] the unknowns are

$$x_1 = -\frac{f(x_0)}{f'(x_0)} \tag{A7}$$

$$x_2 = -\frac{f''(x_0)}{f'(x_0)}\left(\frac{f(x_0)}{f'(x_0)}\right)^2 \tag{A8}$$

$$x_3 = \frac{1}{f'(x_0)}\left(\frac{1}{6}\left(\frac{f(x_0)}{f'(x_0)}\right)^3\right)f'''(x_0) + \frac{f(x_0)}{f'(x_0)}\left(-\frac{f''(x_0)}{f'(x_0)}\left(\frac{f(x_0)}{f'(x_0)}\right)^2\right) \tag{A9}$$

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