



STABILITY AND HOPF BIFURCATION ANALYSIS OF SIR EPIDEMIC MODEL WITH TIME DELAY

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ABSTRACT

A delayed SIR epidemic model in which the susceptible are assumed to satisfy the logistic equation will be taken up for detailed study. The locally asymptotical stability of the disease-free equilibrium and endemic equilibrium will be studied. Further, the Hopf bifurcation analysis will also be addressed. Also, the theoretical analysis will be supported by Numerical simulations for different parametric values.

Keywords: SIR model, logistic growth rate, basic reproduction number, stability, Hopf bifurcation.

1. INTRODUCTION

Mathematical modelling in epidemiology provides us with an understanding of the mechanisms that impact and influence the spread of diseases and in the process, advances the possibilities for control strategies.

To start with, the study takes into account a population that is divided into three types: susceptible, infective and recovered. Let $S(t)$, $I(t)$ and $R(t)$ be the number of susceptible, infective and recovered individuals at time t .

The terminology that comes in handy to discuss and develop the concept advanced is SIR- a concept that elegantly describes a disease that promises immunity against re-infection and indicates that the passage of individuals is from the susceptible class S to the infective class I and then to the removed class R .

Kermack-MC Kendrick proposed a classical SIR epidemic model in 1927. i.e.

$$\begin{aligned}\frac{ds}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI - \alpha I \\ \frac{dR}{dt} &= \alpha I\end{aligned}$$

Where $S(t)$, $I(t)$, $R(t)$ represent the number of susceptible, infective and recovered individuals respectively at time t . The parameters β , α are transmission rates and recovery rates respectively.

The interpretation of the above model is based on the interaction of the population in question. Infectives are instrumental in the decrease of susceptible per unit time. After Kermack-MC Kendrick model, different epidemic models have been proposed by investigators such as

Hethcote [3] and Tudor, Ruan and wang, Derrick, Van den Driesche and R. M. Anderson and R. M. May [6].

In mathematical epidemiology, an important theme that is popular is one that is related to basic reproduction [10]. To begin with, one has number R_0 which serves as a threshold parameter that determines the spread of infectious diseases in a population. R_0 is defined as the average number of secondary infections produced when a single infected individual is introduced into a susceptible population. When $R_0 > 1$, the disease can permeate a totally susceptible population and the number cases will register an increase as a consequence, while on the other hand when $R_0 < 1$, the disease will fail to spread. In SIR model the basic reproduction number determines whether there will be an epidemic.

This paper argues that susceptible individuals are assumed to have the logistic growth with a carrying capacity K ($K > 0$) as well as intrinsic birth rate r ($r > 0$) and where the incidence term is of bilinear mass action. It is apparent that time delays can exercise a role that is both complex and intriguing on the dynamic behaviour of a system [1, 2, 4, 5, 7, 8]. They can have an adverse impact on the stability of a system by creating oscillations and chaos phenomena. It is common knowledge that studies on dynamical systems not only involve stability but also involve several other kinds of behaviour, such as persistence, attractiveness and periodic solution [11]. In particular the properties of periodic solutions carry significant interest to any explorer. Based on this consideration, the inclusion of time delay into susceptible and infective individuals in transmission rate, although only in the first equation becomes imperative since susceptible individuals infected at time $t - \tau$ are expected to just as they are able to spread the disease at time t as well.

The rest of the work is organized as follows. It ferrets out all mathematically conceivable equilibrium points and their stability analysis in sections 3 and 4. We study the existence of the Hopf bifurcation at the positive equilibrium in section 4. Numerical simulations are



performed in section 5. Finally, our conclusions are in section 6.

2. MATHEMATICAL MODEL

In this paper, we shall consider the following delay differential equations

$$\begin{aligned}\frac{dS}{dt} &= rS \left(1 - \frac{S}{k}\right) - \beta S(t-\tau)I(t-\tau) + \mu S \\ \frac{dI}{dt} &= \beta SI - dI\end{aligned}\quad (1)$$

where $S(t)$, $I(t)$ represents the number of susceptible and infected population respectively. And 'r' represents intrinsic birth rate constant, 'k' represents Carrying capacity of susceptible, ' β ' represents the force of infection or the rate of transmission, ' μ ' represents Immigration coefficient of $S(t)$, 'd' represents death coefficient of $I(t)$ and τ is time delay.

3. EQUILIBRIUM ANALYSIS

Now we investigate the existence of equilibria of system (1). System (1) has always a disease-free

equilibrium $E_0 \left(\frac{k}{r}(r+\mu), 0 \right)$ and an endemic

equilibrium $E_1 \left(\frac{d}{\beta}, \frac{k\beta(r+\mu)-rd}{k\beta^2} \right)$.

The basic reproduction number for the model is

$$R_0 = \frac{k\beta(r+\mu)}{rd} \quad (2)$$

4. LOCAL STABILITY ANALYSIS

In this section, we shall investigate the stability analysis of disease-free equilibrium E_0 and endemic equilibrium E_1 . Let $u_1(t)$, $u_2(t)$ be small perturbations for $S(t)$, $I(t)$ respectively, i.e. by considering $S(t) = S^* + u_1(t)$, $I(t) = I^* + u_2(t)$, and by linearizing (1) we get

$$\begin{aligned}\frac{du_1}{dt} &= \left(r + \mu - \frac{2rS^*}{k} \right) u_1 - \beta I^* u_1(t-\tau) - \beta S^* u_2(t-\tau) \\ \frac{du_2}{dt} &= \beta I^* u_1 + (\beta S^* - d) u_2\end{aligned}\quad (3)$$

$$\begin{aligned}P_1 &= -r - \mu + \frac{2rS^*}{k} - \beta S^* + d, P_2 = r\beta S^* - rd + \mu\beta S^* - \mu d - \frac{2r\beta S^{*2}}{k} + \frac{2rdS^*}{k}, \\ Q_1 &= \beta I^*, Q_2 = \beta d I^*\end{aligned}$$

4.1. Disease-Free equilibria and stability

Theorem 1: The disease-free equilibrium is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof: For the disease free equilibrium E_0 , the system (3) reduces to

$$\begin{bmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{bmatrix} = \begin{bmatrix} r + \mu - \frac{2rS^*}{k} - \lambda & -\beta e^{-\lambda\tau} S^* \\ 0 & \beta S^* - d - \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (4)$$

with characteristic equation

$$\left(r + \mu - \frac{2rS^*}{k} - \lambda \right) (\beta S^* - d - \lambda) = 0 \quad (5)$$

The characteristic roots are given by

$$\lambda_1 = -(r + \mu), \lambda_2 = \frac{k\beta(r + \mu) - rd}{r}, \quad \text{then the}$$

system is stable if $k\beta(r + \mu) < rd$ i.e., $R_0 < 1$.

Hence, given system is stable if $R_0 < 1$, and unstable if $R_0 > 1$

4.2. Endemic equilibria and stability

For the endemic equilibrium E_1 , the system (3) reduces to

$$\begin{bmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{bmatrix} = \begin{bmatrix} r + \mu - \frac{2rS^*}{k} - \beta I^* e^{-\lambda\tau} - \lambda & -\beta e^{-\lambda\tau} S^* \\ \beta I^* & \beta S^* - d - \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (6)$$

The characteristic equation of (6) for the endemic equilibrium is

$$\lambda^2 + P_1\lambda + P_2 + e^{-\lambda\tau} (Q_1\lambda + Q_2) = 0 \quad (7)$$

where



We need to find the necessary and sufficient condition for every root of the characteristic equation (6) having negative real part.

Case1: For $\tau = 0$, (6) becomes

$$\lambda^2 + (P_1 + Q_1)\lambda + (P_2 + Q_2) = 0 \quad (7)$$

By Routh-Hurwitz criteria, all roots of (7) are real and negative or complex conjugate with negative real part if $P_1 + Q_1 > 0$ & $P_2 + Q_2 > 0$.

Hence the system (1) without delay is locally asymptotically stable when $R_0 > 1$.

Case2: If $\tau > 0$

Put $\lambda = i\omega$ in (6), we get

$$-\omega^2 + P_1\omega i + P_2 + (Q_1\omega i + Q_2)[\cos \omega\tau - i \sin \omega\tau] = 0 \quad (8)$$

Separating the real and imaginary parts, we get

$$\begin{aligned} \omega^2 - P_2 &= Q_1\omega \sin \omega\tau + Q_2 \cos \omega\tau \\ -P_1\omega &= -Q_2 \sin \omega\tau + Q_1\omega \cos \omega\tau \end{aligned} \quad (9)$$

which is equivalent to

$$\omega^4 + (P_1^2 - 2P_2 - Q_1^2)\omega^2 + (P_2^2 - Q_2^2) = 0 \quad (10)$$

Thus, if $P_1^2 - 2P_2 - Q_1^2 > 0, P_2^2 - Q_2^2 > 0$ then there is no ω such that $i\omega$ is an Eigen value of the characteristic equation (6) i.e., λ will never be a purely imaginary root of equation (6). Thus the real parts of all Eigen values of (6) are negative for all $\tau \geq 0$. Hence endemic equilibrium E_1 is asymptotically stable for all τ if the following conditions hold:

- i). $R_0 > 1$
- ii). $(P_1 + Q_1) > 0, (P_2 + Q_2) > 0$
- iii). $P_1^2 - 2P_2 - Q_1^2 > 0, P_2^2 - Q_2^2 > 0$

If any one of $P_1^2 - 2P_2 - Q_1^2, P_2^2 - Q_2^2$ is negative, there is a unique positive ω_0 satisfying (10). That is there is a single pair of purely imaginary roots $\pm i\omega_0$ to (6).

From (9) τ_k corresponding to ω_0 can be obtained

$$\tau_k = \frac{1}{\omega_0} \arccos \left[\frac{(Q_2 - P_1 Q_1) \omega^2 - P_2 Q_2}{Q_1^2 \omega^2 + Q_2^2} \right] + \frac{2n\pi}{\omega_0}, n = 0, 1, 2, \dots \quad (12)$$

For $\tau = 0, E_1$ is stable, it remains stable for

$$\tau < \tau_0 \text{ if } \left. \frac{d \operatorname{Re}(\lambda)}{dt} \right|_{\lambda=i\omega_0} > 0.$$

Differentiating (6) with respect τ , we get

$$\frac{d\lambda}{d\tau} [2\lambda + P_1 + Q_1 e^{-\lambda\tau} - (Q_1\lambda + Q_2)\tau e^{-\lambda\tau}] = \lambda(Q_1\lambda + Q_2)e^{-\lambda\tau} \quad (13)$$

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda + P_1 + Q_1 e^{-\lambda\tau} - (Q_1\lambda + Q_2)\tau e^{-\lambda\tau}}{\lambda(Q_1\lambda + Q_2)e^{-\lambda\tau}}$$

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda + P_1}{\lambda(Q_1\lambda + Q_2)e^{-\lambda\tau}} + \frac{Q_1}{\lambda(Q_1\lambda + Q_2)} - \frac{\tau}{\lambda}$$

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda + P_1}{-\lambda(\lambda^2 + P_1\lambda + P_2)} + \frac{Q_1}{\lambda(Q_1\lambda + Q_2)} - \frac{\tau}{\lambda}$$

$$\left. \frac{d \operatorname{Re}(\lambda)}{d\tau} \right|_{\lambda=i\omega_0} = \operatorname{Re} \left(\left. \frac{d\lambda}{d\tau} \right|_{\lambda=i\omega_0} \right)^{-1}$$

$$= \operatorname{Re} \left[\frac{2i\omega_0 + P_1}{-i\omega_0(-\omega_0^2 + P_1\omega_0 + P_2)} + \frac{Q_1}{i\omega_0(Q_1i\omega_0 + Q_2)} - \frac{\tau}{i\omega_0} \right]$$

$$= \operatorname{Re} \left[\frac{1}{\omega_0} \left(\frac{2i\omega_0 + P_1}{P_1\omega_0 + (\omega_0^2 - P_2)i} + \frac{Q_1}{(-Q_1\omega_0 + Q_2)i} + \tau i \right) \right]$$

$$= \frac{1}{\omega_0} \left(\frac{2\omega_0(\omega_0^2 - P_2) + P_1^2\omega_0}{P_1^2\omega_0 + (\omega_0^2 - P_2)^2} - \frac{Q_1^2}{(Q_1^2\omega_0^2 + Q_2^2)} \right)$$

$$= \frac{2\omega_0^2 + (P_1^2 - 2P_2 - Q_1^2)}{(Q_1^2\omega_0^2 + Q_2^2)}$$

Under the condition $P_1^2 - 2P_2 - Q_1^2 > 0$, we

$$\text{have } \left. \frac{d \operatorname{Re}(\lambda)}{d\tau} \right|_{\lambda=i\omega_0} > 0.$$

Therefore, the transversality condition holds and Hopf bifurcation occurs at $\omega = \omega_0, \tau = \tau_0$.

5. NUMERICAL SIMULATION

In this section, we present some numerical results of system (1) to verify the analytical predications obtained in the previous section. We take the parameter values of the system as

$\beta = 0.1, r = 10, k = 10, \mu = 0.78, d = 0.95$, which has a positive equilibrium as $E_1(9.5, 12.8)$ &



$R_0 = 1.1347$ and satisfies conditions indicated in theorem (1). We can obtain positive time delay $\tau_0 = 11.416$. Thus we know that $0 \leq \tau < \tau_0$, E_1 is asymptotically stable. When τ passes through the critical

value τ_0 , $E_1(9.5, 12.8)$ loses its stability and a Hopf bifurcation occurs and a family of periodic solutions bifurcate from E_1 , which can be illustrated in Figures 1-3.

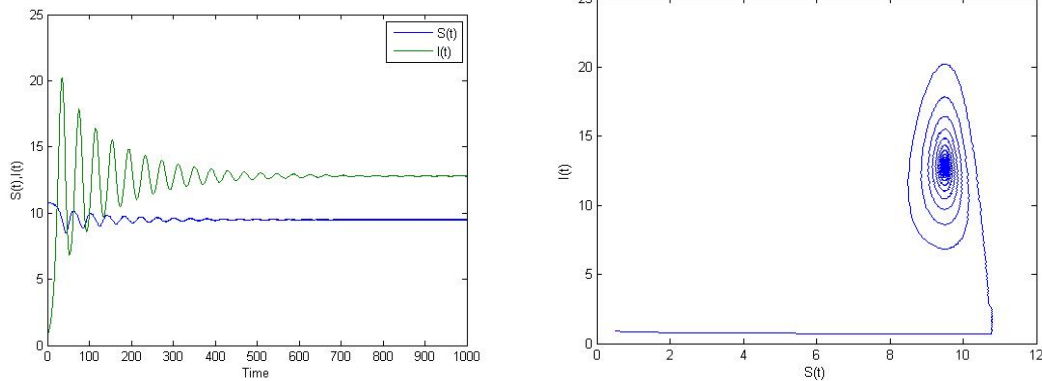


Figure-1. The trajectories and graphs of system (1) with $\tau = 10.5 < \tau_0 = 11.416$

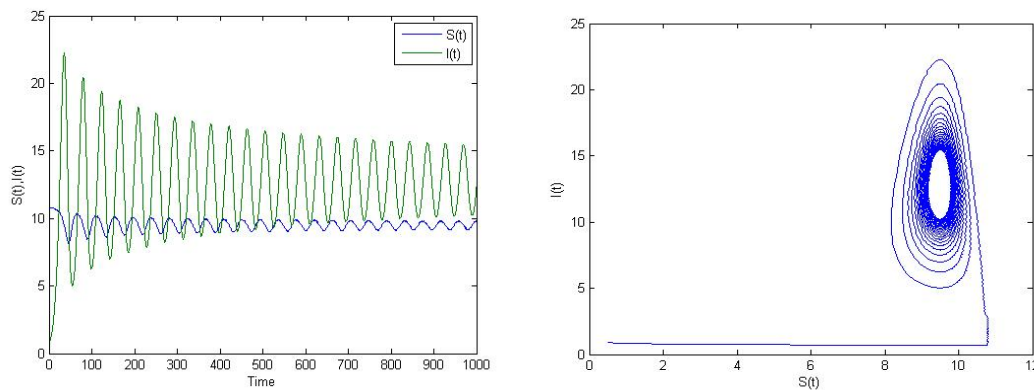


Figure-2. The trajectories and phase graphs of system (1) with $\tau = \tau_0 = 11.416$

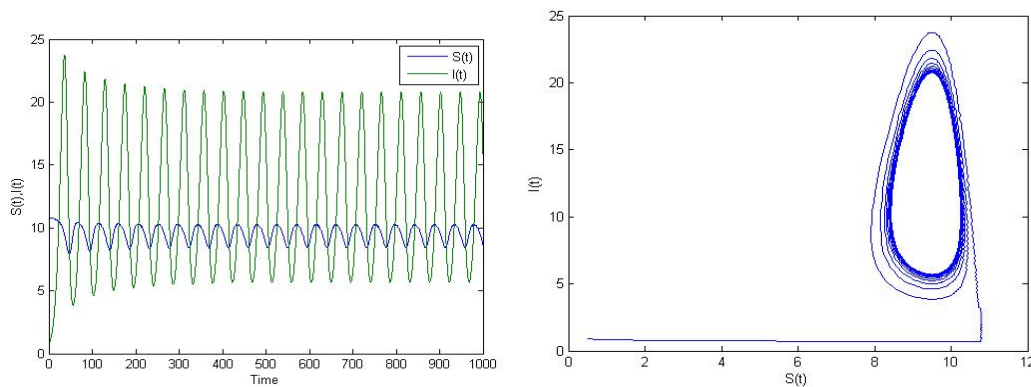


Figure-3. The trajectories and phase graphs of system (1) with $\tau = 12 > \tau_0 = 11.416$



6. CONCLUSIONS

In this paper we have considered a SIR model with time delay and the susceptible follow the logistic growth. The global dynamical behaviour of the model is studied and the threshold value R_0 of the system is defined which determines the behaviours of the system. If $R_0 < 1$, the disease free equilibrium E_0 is asymptotically stable and if $R_0 > 1$ the endemic equilibrium E_1 is asymptotically stable. The system changes its behaviour from stable to unstable nature around E_1 when τ crosses τ_0 , the equilibrium loses its stability and Hopf bifurcation occur at E_1 and periodic orbits bifurcating from E_1 . The numerical simulations performed illustrate our theoretical results.

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