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OPTIMAL BOUNDARY CONTROL FOR THE INCOMPRESSIBLE VISCOELASTIC FLUID SYSTEM

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ABSTRACT

We consider the boundary control problem for the nonlinear system describing steady flows of viscoelastic fluids of the Jeffreys kind in a bounded domain of R^d (d=2 or 3). We study the existence of a weak solution minimizing a given cost functional. To prove the solvability of the control problem we construct a sequence of finite-dimensional approximating problems. For approximate solutions we obtain a priori estimates, which do not depend on the parameters of the approximation. This allows us to select a subsequence of approximate solutions that converges to some weak solution of the original problem. Using properties of the weak solutions and the cost functional, we obtain some sufficient conditions for the existence of optimal solutions.

Keywords: optimal control, boundary control, the Jeffreys model, weak solutions.

1. INTRODUCTION

Boundary control problems for nonlinear equations of hydrodynamics have always great attention of specialists in control theory and applications (see, e.g., [1-7] and the references therein). In this paper we consider boundary control for the steady motion equations of viscoelastic fluids of the Jeffreys kind [8]. Note that the controllability properties of Jeffreys fluids were first analyzed in [7]. Some results for the corresponding optimal control problem have been established in [9]. Optimal boundary control in the Jeffreys model has not yet been studied specifically. The aim of the present paper is to study such problem.

The paper is organized as follows. In Section 2, the necessary information is given on the Jeffreys model and the corresponding function spaces. In Section 3, we study the solvability and some properties of solutions to the Dirichlet boundary value problem for the Jeffreys model. In Section 4, the optimal boundary control problem is formulated for the stationary equations of the Jeffreys model in a bounded domain of R^d (d=2 or 3). Based on the results from Section 3 we establish the conditions under which the control system is solvable and has an optimal solution.

2. PRELIMINARIES

The movement of an incompressible viscoelastic fluid of Jeffreys type described by the following equations:

$$\rho(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) + \nabla p - \operatorname{Div}\mathbf{S} = \rho\mathbf{g},\ \ \nabla\cdot\mathbf{v} = 0, \tag{1}$$

$$\mathbf{S} + \lambda_1 \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{S} = 2\mu_0(\mathbf{D}(\mathbf{v}) + \lambda_2 \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}(\mathbf{v})). \tag{2}$$

Here $\mathbf{v}$ is the velocity, $\mathbf{D}(\mathbf{v})$ is the strain velocity tensor, $\mathbf{D}(\mathbf{v}) = (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)/2$, p is the pressure, $\mathbf{g}$ is the body force, $\mathbf{S}$ is the extra-stress, $\mathrm{d}/\mathrm{d}t$ is the material time derivative, ρ is the density (for simplicity, we will assume in what follows that $\rho = 1$), μ_0 is the viscosity, λ_1 is the relaxation time, λ_2 is the retardation time ($\mu_0 > 0$, $0 < \lambda_2 < \lambda_1$).

Boundary and initial-boundary value problems for liquids described by system (1), (2) and other similar non-Newtonian models were studied by many authors (see [10–20]).

We need certain function spaces (more details can be found, for example, in [21]). Denote by $M_s^{d\times d}$ the space of $d\times d$ symmetric matrices. Let Ω be a bounded, locally Lipschitz domain in space R^d, and let F denote one of the spaces R, R^d, and $M_s^{d\times d}$. We shall use the standard notations $L^p(\Omega,F)$, $H^m(\Omega,F)$ for Lebesgue and Sobolev spaces of functions from Ω into F. We denote the scalar product in the space $L^2(\Omega,F)$ by $(\cdot,\cdot)$.

Let $C_0^\infty(\Omega,F)$ be the space of C^∞-smooth functions $\mathbf{v}:\Omega\to F$ such that $\operatorname{supp}\mathbf{v}\subset\Omega$. Let

$$Y(\Omega,R^d) = \{\mathbf{v}\in C_0^\infty(\Omega,R^d) : \nabla\cdot\mathbf{v} = 0\},\ Z(\Omega,R^d) = \{\mathbf{v}\in H^1(\Omega,R^d) : \nabla\cdot\mathbf{v} = 0\}.$$

By $H_0^m(\Omega,F)$ denote the closure of $C_0^\infty(\Omega,F)$ in $H^m(\Omega,F)$. Let V be the closure of $Y(\Omega,R^d)$ in $H_0^1(\Omega,R^d)$.

We define the norms in the spaces $Z(\Omega,R^d)$ and V by the following formulas

$$\|\mathbf{v}\|_{Z(\Omega,R^d)} = \left(\|\mathbf{D}(\mathbf{v})\|^2_{L^2(\Omega,M_s^{d\times d})} + \|\mathbf{v}\|^2_{L^2(\partial\Omega,R^d)}\right)^{1/2},\ \|\mathbf{v}\|_V = \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega,M_s^{d\times d})}.$$

Here the restriction of $\mathbf{v}$ to the boundary $\partial\Omega$ is given by the formula $\mathbf{v}|_{\partial\Omega} = \gamma_0\mathbf{v}$, where $\gamma_0 : H^1(\Omega,R^d)\to H^{1/2}(\partial\Omega,R^d)$ is the trace operator.

By $\dot{H}^{1/2}(\partial\Omega, R^d)$ denote the space of function $\mathbf{w} \in H^{1/2}(\partial\Omega, R^d)$ such that $\int_{\partial\Omega} \mathbf{w}\cdot\mathbf{n} = 0.$

For a Banach space B, the symbol B^* denotes the conjugate space. The value of a functional $\mathbf{f} \in B^*$ on an element $\mathbf{v} \in B$ is denoted by $\langle \mathbf{f}, \mathbf{v}\rangle$.

3. AUXILIARY RESULTS

Consider the boundary value problem describing steady flows of Jeffreys fluids in the domain Ω:

$$\mathbf{v}\cdot\nabla\mathbf{v} + \nabla p - \mathrm{Div}\,\mathbf{S} = \mathbf{g},\ \ \nabla\cdot\mathbf{v} = 0,\ \ \mathbf{x}\in\Omega, \tag{3}$$

$$\mathbf{S} + \lambda_1 \mathbf{v}\cdot\nabla\mathbf{S} = 2\mu_0(\mathbf{D}(\mathbf{v}) + \lambda_2 \mathbf{v}\cdot\nabla\mathbf{D}(\mathbf{v})),\ \ \mathbf{x}\in\Omega, \tag{4}$$

$$\mathbf{v}|_{\partial\Omega} = \mathbf{u}. \tag{5}$$

Following [12], we introduce the concept of a weak solution. Let us suppose that $\mathbf{g} \in V^*$ and $\mathbf{u} \in \dot{H}^{1/2}(\partial\Omega, R^d)$.

Definition. A *weak solution* of problem (3)–(5) is a pair of functions $(\mathbf{v}, \mathbf{S}) \in Z(\Omega, R^d) \times L^2(\Omega, M_s^{d\times d})$ such that $\gamma_0 \mathbf{v} = \mathbf{u}$ and

$$-\sum_{i=1}^{d}(v_i\mathbf{v}, \frac{\partial\boldsymbol{\varphi}}{\partial x_i}) + (\mathbf{S}, \mathbf{D}(\boldsymbol{\varphi})) = \langle \mathbf{g}, \boldsymbol{\varphi}\rangle, \tag{6}$$

$$(\mathbf{S}, \boldsymbol{\Psi}) - \lambda_1 \sum_{i=1}^{d}(v_i\mathbf{S}, \frac{\partial\boldsymbol{\Psi}}{\partial x_i}) = -2\mu_0(\mathbf{v}, Div\boldsymbol{\Psi}) - 2\mu_0\lambda_2 \sum_{i=1}^{d}(v_i\mathbf{D}(\mathbf{v}), \frac{\partial\boldsymbol{\Psi}}{\partial x_i}) \tag{7}$$

for any $\boldsymbol{\varphi} \in Y(\Omega, R^d)$ and $\boldsymbol{\Psi} \in C_0^\infty(\Omega, M_s^{d\times d})$.

The existence of weak solutions is proved in [12, 13]. The following theorem is needed for the sequel.

Theorem-1. Assume $\mathbf{u} = \mathbf{0}$. Then problem (3)–(5) has at least one weak solution satisfying the estimate

$$\|\mathbf{D}(\mathbf{v})\|^2_{L^2(\Omega, M_s^{d\times d})} + \frac{\lambda_1^2}{4\mu_0\lambda_2(\lambda_1-\lambda_2)}\|\mathbf{S} - \frac{2\mu_0\lambda_2}{\lambda_1}\mathbf{D}(\mathbf{v})\|^2_{L^2(\Omega, M_s^{d\times d})} \le \frac{\lambda_1^2}{4\mu_0^2\lambda_2^2}\|\mathbf{g}\|^2_{V^*}.$$

Proof. Let us introduce the notations $\mu_1 = \mu_0\lambda_2/\lambda_1,\ \mu_2 = (\mu_0 - \mu_1)/\lambda_1,\ \mathbf{E} = \mathbf{S} - 2\mu_1\mathbf{D}(\mathbf{v})$.

Then equations (6) and (7) may be rewritten as

$$-\sum_{i=1}^{d}(v_i\mathbf{v}, \frac{\partial\boldsymbol{\varphi}}{\partial x_i}) + (\mathbf{E}, \mathbf{D}(\boldsymbol{\varphi})) + 2\mu_1(\mathbf{D}(\mathbf{v}), \mathbf{D}(\boldsymbol{\varphi})) = \langle \mathbf{g}, \boldsymbol{\varphi}\rangle, \tag{8}$$

$$\frac{1}{\lambda_1}(\mathbf{E}, \boldsymbol{\Psi}) - 2\mu_2(\mathbf{D}(\mathbf{v}), \boldsymbol{\Psi}) - \sum_{i=1}^{d}\left(v_i\mathbf{E}, \frac{\partial\boldsymbol{\Psi}}{\partial x_i}\right) = 0. \tag{9}$$

Let $\boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2, \ldots$ be an orthonormal basis of the space V, and let $\boldsymbol{\Psi}^1, \boldsymbol{\Psi}^2, \ldots$ be an orthonormal basis of $H_0^2(\Omega, M_s^{d\times d})$ such that $\boldsymbol{\Psi}^j \in C_0^\infty(\Omega, M_s^{d\times d})$. Let us fix $m \in N$. Consider the auxiliary problem:
Find a pair of functions $(\mathbf{v}^m, \mathbf{E}^m)$ such that

$$-\theta\sum_{i=1}^{d}\left(v_i^m\mathbf{v}^m, \frac{\partial\boldsymbol{\varphi}^j}{\partial x_i}\right) + \theta(\mathbf{E}^m, \mathbf{D}(\boldsymbol{\varphi}^j)) + 2\mu_1(\mathbf{D}(\mathbf{v}^m), \mathbf{D}(\boldsymbol{\varphi}^j)) = \theta\langle \mathbf{g}, \boldsymbol{\varphi}^j\rangle,\ j = 1\ldots m, \tag{10}$$

$$\frac{1}{\lambda_1}(\mathbf{E}^m, \boldsymbol{\Psi}^j) - 2\mu_2\theta(\mathbf{D}(\mathbf{v}^m), \boldsymbol{\Psi}^j) + \frac{1}{m}(\Delta\mathbf{E}^m, \Delta\boldsymbol{\Psi}^j) + \theta\sum_{i=1}^{d}\left(\frac{\partial\mathbf{E}^m}{\partial x_i}, v_i^m\boldsymbol{\Psi}^j\right) = 0,\ j = 1\ldots m, \tag{11}$$

$$\mathbf{v}^m = \sum_{j=1}^{m} a_{mj}\boldsymbol{\varphi}^j,\ \ \mathbf{E}^m = \sum_{j=1}^{m} b_{mj}\boldsymbol{\Psi}^j, \tag{12}$$

where a_{mj} and b_{mj} are unknown, θ is a parameter, and $\theta \in [0,1]$.

Let us first establish a priori estimates for solutions of problem (10)–(12). Let a pair $(\mathbf{v}^m, \mathbf{E}^m)$ satisfy (10)–(12). We multiply (10) by a_{mj} and sum up the corresponding equalities over j from 1 to *m*. The result is

$$\theta(\mathbf{E}^m, \mathbf{D}(\mathbf{v}^m)) + 2\mu_1(\mathbf{D}(\mathbf{v}^m), \mathbf{D}(\mathbf{v}^m)) = \theta\langle \mathbf{g}, \mathbf{v}^m\rangle. \tag{13}$$

Now we multiply (11) by b_{mj} and sum up the corresponding equalities over j from 1 to *m*. The result is

$$\frac{1}{\lambda_1}(\mathbf{E}^m, \mathbf{E}^m) - 2\mu_2\theta(\mathbf{D}(\mathbf{v}^m), \mathbf{E}^m) + \frac{1}{m}(\Delta\mathbf{E}^m, \Delta\mathbf{E}^m) = 0. \tag{14}$$

Multiply (14) by $1/(2\mu_2)$ and sum with (13). The result is

$$2\mu_1(\mathbf{D}(\mathbf{v}^m), \mathbf{D}(\mathbf{v}^m)) + \frac{1}{2\mu_2\lambda_1}(\mathbf{E}^m, \mathbf{E}^m) + \frac{1}{2\mu_2 m}(\Delta\mathbf{E}^m, \Delta\mathbf{E}^m) = \theta\langle \mathbf{g}, \mathbf{v}^m\rangle.$$

It now follows that

$$2\mu_1\|\mathbf{D}(\mathbf{v}^m)\|^2_{L^2(\Omega, M_s^{d\times d})} + \frac{1}{2\mu_2\lambda_1}\|\mathbf{E}^m\|^2_{L^2(\Omega, M_s^{d\times d})} + \frac{1}{2\mu_2 m}\|\mathbf{E}^m\|^2_{H_0^2(\Omega, M_s^{d\times d})} \le \|\mathbf{g}\|_{V^*}\|\mathbf{D}(\mathbf{v}^m)\|_{L^2(\Omega, M_s^{d\times d})}. \tag{15}$$

This yields that

$$\|\mathbf{D}(\mathbf{v}^m)\|_{L^2(\Omega, M_s^{d\times d})} \le \frac{1}{2\mu_1}\|\mathbf{g}\|_{V^*}. \tag{16}$$

It follows from (15) and (16) that

$$2\mu_1\|\mathbf{D}(\mathbf{v}^m)\|^2_{L^2(\Omega, M_s^{d\times d})} + \frac{1}{2\mu_2\lambda_1}\|\mathbf{E}^m\|^2_{L^2(\Omega, M_s^{d\times d})} + \frac{1}{2\mu_2 m}\|\mathbf{E}^m\|^2_{H_0^2(\Omega, M_s^{d\times d})} \le \frac{1}{2\mu_1}\|\mathbf{g}\|^2_{V^*}. \tag{17}$$

Note that this estimate is independent of θ. Using the methods of topological degree theory (see [22]), we can easily establish the solvability of problem (10)–(12) for any $\theta \in [0,1]$.

Fix $\theta = 1$. Let $(\mathbf{v}^m, \mathbf{E}^m)$ be a sequence of solutions of (10)–(12) for $m = 1,2,\ldots$. It follows from (17) that the norms $\|\mathbf{v}^m\|_V$, $\|\mathbf{E}^m\|_{L^2(\Omega, M_s^{d\times d})}$ are majorized by a constant that does not depend on *m*. Hence, we can assume without loss of generality that $\mathbf{v}^m \to \mathbf{v}^*$ weakly in V, $\mathbf{E}^m \to \mathbf{E}^*$ weakly in $L^2(\Omega, M_s^{d\times d})$ as $m \to \infty$. Note that the embedding $H^1(\Omega, R^d) \subset L^4(\Omega, R^d)$ is compact (see e.g. [21]). Thus, $\mathbf{v}^m \to \mathbf{v}^*$ strongly in $L^4(\Omega, R^d)$. Therefore, we can pass to the limit $m \to \infty$ in equality (3.8) (with $\theta = 1$) and obtain

$$-\sum_{i=1}^{d}\left(v_i^*\mathbf{v}^*, \frac{\partial \boldsymbol{\varphi}^j}{\partial x_i}\right) + (\mathbf{E}^*, \mathbf{D}(\boldsymbol{\varphi}^j)) + 2\mu_1(\mathbf{D}(\mathbf{v}^*), \mathbf{D}(\boldsymbol{\varphi}^j)) = \langle \mathbf{g}, \boldsymbol{\varphi}^j \rangle \tag{18}$$

for each $j \in N$. Since $\{\boldsymbol{\varphi}^j\}$ is a basis of the space V, equality (18) remains valid if we replace $\boldsymbol{\varphi}^j$ with an arbitrary function $\boldsymbol{\varphi} \in V$.

Further, transforming the third and fourth terms in (11) (with $\theta = 1$) by the integration by parts, we obtain

$$\frac{1}{\lambda_1}\left(\mathbf{E}^m, \boldsymbol{\Psi}^j\right) - 2\mu_2(\mathbf{D}(\mathbf{v}^m), \boldsymbol{\Psi}^j) + \frac{1}{m}(\mathbf{E}^m, \Delta^2\boldsymbol{\Psi}^j) - \sum_{i=1}^{d}\left(\mathbf{E}^m, v_i^m \frac{\partial \boldsymbol{\Psi}^j}{\partial x_i}\right) = 0.$$

We can take the limit in this equation as $m \to \infty$, and obtain:

$$\frac{1}{\lambda_1}\left(\mathbf{E}^*, \boldsymbol{\Psi}^j\right) - 2\mu_2(\mathbf{D}(\mathbf{v}^*), \boldsymbol{\Psi}^j) - \sum_{i=1}^{d}\left(\mathbf{E}^*, v_i^* \frac{\partial \boldsymbol{\Psi}^j}{\partial x_i}\right) = 0. \tag{19}$$

It is obvious that equality (19) remains valid if we replace the function $\boldsymbol{\Psi}^j$ with $\boldsymbol{\Phi} \in C_0^\infty(\Omega, M_s^{d\times d})$. Consequently, the pair $(\mathbf{v}^*, \mathbf{E}^*)$ satisfies equations (8) and (9). It is readily seen that the pair $(\mathbf{v}^*, \mathbf{E}^* + 2\mu_1\mathbf{D}(\mathbf{v}^*))$ is a weak solution to problem (3), (4) with the boundary condition $\mathbf{u} = \mathbf{0}$. Using (17), we get the following the estimate

$$\|\mathbf{D}(\mathbf{v}^*)\|^2_{L^2(\Omega, M_s^{d\times d})} + \frac{\lambda_1^2}{4\mu_0\lambda_2(\lambda_1 - \lambda_2)}\|\mathbf{S}^* - \frac{2\mu_0\lambda_2}{\lambda_1}\mathbf{D}(\mathbf{v}^*)\|^2_{L^2(\Omega, M_s^{d\times d})} \le \frac{\lambda_1^2}{4\mu_0^2\lambda_2^2}\|\mathbf{g}\|^2_{V^*},$$

where $\mathbf{S}^* = \mathbf{E}^* + 2\mu_1\mathbf{D}(\mathbf{v}^*)$. This completes the proof of Theorem 1.

4. MAIN RESULTS

Consider the following optimal control problem

$$\mathbf{v}\cdot\nabla\mathbf{v} + \nabla p - \operatorname{Div}\mathbf{S} = \mathbf{g},\ \nabla\cdot\mathbf{v} = 0,\ \mathbf{x} \in \Omega, \tag{20}$$

$$\mathbf{S} + \lambda_1\mathbf{v}\cdot\nabla\mathbf{S} = 2\mu_0(\mathbf{D}(\mathbf{v}) + \lambda_2\mathbf{v}\cdot\nabla\mathbf{D}(\mathbf{v})),\ \mathbf{x} \in \Omega, \tag{21}$$

$$\mathbf{v}|_{\partial\Omega} = \mathbf{u}, \tag{22}$$

$$(\mathbf{v}, \mathbf{u}) \in W \times U, \tag{23}$$

$$J(\mathbf{v}, \mathbf{S}, \mathbf{u}) \to \min, \tag{24}$$

where $\mathbf{u}$ is the control, U is the set of admissible controls $\mathbf{u}$, W is the set of admissible states $\mathbf{v}$, J is the cost functional. We suppose for simplicity that

$$W = \{\mathbf{v} \in Z(\Omega, R^d) : \|\mathbf{v}\|_{Z(\Omega, R^d)} \le R\},$$

where *R* is a constant.

Definition. A triplet $(\mathbf{v}, \mathbf{S}, \mathbf{u}) \in Z(\Omega, R^d) \times L^2(\Omega, M_s^{d\times d}) \times H^{1/2}(\partial\Omega, R^d)$ is called a *weak solution* of problem (20)–(23) if $(\mathbf{v}, \mathbf{u}) \in W \times U$ and the pair $(\mathbf{v}, \mathbf{S})$ is a weak solution of problem (20)–(22).

Denote the set of weak solutions of problem (20)–(23) by *K*.

Definition. A triplet $(\mathbf{v}_0, \mathbf{S}_0, \mathbf{u}_0) \in K$ is called an *optimal solution* of problem (4.1)–(4.5) if $J(\mathbf{v}_0, \mathbf{S}_0, \mathbf{u}_0) = \inf_{(\mathbf{v}, \mathbf{S}, \mathbf{u}) \in K} J(\mathbf{v}, \mathbf{S}, \mathbf{u})$.

We assume that the cost functional $J : Z(\Omega, R^d) \times L^2(\Omega, M_s^{d\times d}) \times L^2(\partial\Omega, R^d) \to R$ satisfies the following conditions:

(i) the set $\{J(\mathbf{v}, \mathbf{S}, \mathbf{u}) : (\mathbf{v}, \mathbf{S}, \mathbf{u}) \in Z(\Omega, R^d) \times L^2(\Omega, M_s^{d\times d}) \times L^2(\partial\Omega, R^d)\}$ is bounded from below;

(ii) for any fixed positive number *r*, the set $\{(\mathbf{v}, \mathbf{S}, \mathbf{u}) \in Z(\Omega, R^d) \times L^2(\Omega, M_s^{d\times d}) \times L^2(\partial\Omega, R^d) : J(\mathbf{v}, \mathbf{S}, \mathbf{u}) \le r\}$ is bounded;

(iii) for a sequence $\{(\mathbf{v}^m, \mathbf{S}^m, \mathbf{u}^m)\}$ such that $\mathbf{v}^m \to \mathbf{v}$ weakly in $Z(\Omega, R^d)$, $\mathbf{S}^m \to \mathbf{S}$ weakly in $L^2(\Omega, M_s^{d\times d})$, and $\mathbf{u}^m \to \mathbf{u}$ strongly in $L^2(\partial\Omega, R^d)$, we have $J(\mathbf{v}, \mathbf{S}, \mathbf{u}) \le \liminf_{m\to\infty} J(\mathbf{v}^m, \mathbf{S}^m, \mathbf{u}^m)$.

We also assume that (iv) the set U is closed in the space $L^2(\partial\Omega, R^d)$.

The main result of this paper is the following theorem.

Theorem 2. Let (i)–(iv) be satisfied. Suppose that $\mathbf{0} \in U$ and $R \ge (\lambda_1/2\mu_0\lambda_2)\|\mathbf{g}\|_{V^*}$. Then problem (20) – (24) has at least one optimal solution.

Proof. From Theorem 1 it follows that the set *K* is not empty. Indeed, $(\mathbf{v}_*, \mathbf{S}_*, \mathbf{0}) \in K$, where the pair

$(\mathbf{v}_*, \mathbf{S}_*)$ is a weak solution of problem (20), (21) with the boundary condition $\mathbf{v}|_{\partial\Omega} = \mathbf{0}$.

Let $\{(\mathbf{v}^m, \mathbf{S}^m, \mathbf{u}^m)\}_{m=1}^{\infty} \subset K$ be a sequence such that

$$\lim_{m\to\infty} J(\mathbf{v}^m, \mathbf{S}^m, \mathbf{u}^m) = \inf_{(\mathbf{v},\mathbf{S},\mathbf{u})\in K} J(\mathbf{v}, \mathbf{S}, \mathbf{u}). \tag{25}$$

From (25) and (i) it follows that the sequence $\{J(\mathbf{v}^m, \mathbf{S}^m, \mathbf{u}^m)\}_{m=1}^{\infty}$ is bounded. Using (ii), we see that the sequence $\{(\mathbf{v}^m, \mathbf{S}^m, \mathbf{u}^m)\}$ is bounded in the space $Z(\Omega, R^d) \times L^2(\Omega, M_s^{d\times d}) \times L^2(\partial\Omega, R^d)$. Hence, there exists a subsequence $\{m_k\}$ such that $\mathbf{v}^{m_k} \to \mathbf{v}^0$ weakly in $Z(\Omega, R^d)$, $\mathbf{S}^{m_k} \to \mathbf{S}^0$ weakly in $L^2(\Omega, M_s^{d\times d})$, $\mathbf{u}^{m_k} \to \mathbf{u}^0$ strongly in $L^2(\partial\Omega, R^d)$ as $k \to \infty$. We shall show that $(\mathbf{v}^0, \mathbf{S}^0, \mathbf{u}^0)$ is an optimal solution of problem (20)–(24).

From condition (iv), $\{\mathbf{u}^{m_k}\} \subset U$, and strongly convergence $\mathbf{u}^{m_k} \to \mathbf{u}^0$ it follows that $\mathbf{u}^0 \in U$. Note also that $\mathbf{v}^0 \in W$. Furthermore, it is not hard to prove that the pair $(\mathbf{v}^0, \mathbf{S}^0)$ is a weak solution of problem (20), (21) with the boundary condition $\mathbf{v}|_{\partial\Omega} = \mathbf{u}^0$. This implies that $(\mathbf{v}^0, \mathbf{S}^0, \mathbf{u}^0) \in K$.

Taking into account condition (iii) and (25), we obtain

$$J(\mathbf{v}^0, \mathbf{S}^0, \mathbf{u}^0) \le \liminf_{k\to\infty} J(\mathbf{v}^{m_k}, \mathbf{S}^{m_k}, \mathbf{u}^{m_k}) = \inf_{(\mathbf{v},\mathbf{S},\mathbf{u})\in K} J(\mathbf{v}, \mathbf{S}, \mathbf{u}).$$

Thus, $(\mathbf{v}^0, \mathbf{S}^0, \mathbf{u}^0)$ is an optimal solution of problem (20)-(24). This completes the proof of Theorem 2.

REFERENCES

[1] Fursikov A.V, Imanuvilov O. Yu. 1999. Exact controllability of the Navier-Stokes and Boussinesq equations. Russian Mathematical Surveys. 54(3): 565-618. DOI: 10.1070/RM1999v054n03ABEH000153.

[2] Fursikov A.V. 2012. Flow of a viscous incompressible fluid around a body: Boundary-value problems and minimization of the work of a fluid. Journal of Mathematical Sciences. 180(6): 763-816. DOI: 10.1007/s10958-012-0670-1.

[3] Starovoitov V.N. 2013. Optimal control of a cylinder rotating in a viscous fluid. Journal of Applied and Industrial Mathematics. 7(2): 259-268. DOI: 10.1134/S1990478913020166.

[4] Baranovskii E.S. 2012. Solvability of the stationary optimal control problem for motion equations of second grade fluids. Siberian Electronic Mathematical Reports. 9: 554-560.

[5] Baranovskii E.S. 2014. An optimal boundary control problem for the motion equations of polymer solutions. Siberian Advances in Mathematics. 24(3): 159-168. DOI:10.3103/S105513441403002X

[6] John C., Wachsmuth D. 2009. Optimal Dirichlet boundary control of stationary Navier-Stokes equations with state constraint. Numerical Functional Analysis and Optimization. 30(11): 1309-1338.

[7] Doubova A., Fernandez-Cara E. 2012. On the control of viscoelastic Jeffreys fluids. Systems & Control Letters, 21: 573-579. DOI:10.1016/j.sysconle.2012.02.003.

[8] Reiner M. 1958. Rheology. Berlin: Springer.

[9] Baranovskii E.S. 2014. Optimal control for steady flows of the Jeffreys fluids with slip boundary condition. Journal of Applied and Industrial Mathematics. 8(2): 168-176. DOI: 10.1134/S1990478914020033.

[10] Guillope C., Saut J.C. 1990. Existence results for the flow of viscoelastic fluids with a differential constitutive law. Nonlinear Analysis: Theory, Methods and Applications, pp. 849-869.

[11] Turganbaev, E.M. (1995). Initial-boundary value problems for the equations of a viscoelastic fluid of Oldroyd type. Siberian Mathematical Journal, 36: 389-403.

[12] Vorotnikov D.A. 2004. On the existence of weak stationary solutions of a boundary value problem in the Jeffreys model of the motion of a viscoelastic medium. Izv. Vyssh. Uchebn. Zaved. Mat., 9: 13-17.

[13] Baranovskii E.S. 2013. An inhomogeneous boundary value problem for the stationary motion equations of Jeffreys viscoelastic medium. Journal of Applied and Industrial Mathematics. 7(1): 22-28. DOI: 10.1134/S1990478913010031.

[14] Zhao W. 2011. The global existence of small solutions to the Oldroyd-B model. Chinese Annals of Mathematics, Series B, 32: 215-222. DOI: 10.1007/s11401-011-0636-5.

[15] Roux C. 2014. On flows of viscoelastic fluids of Oldroyd type with wall slip. Journal of Mathematical Fluid Mechanics, 16: 335-350. DOI: 10.1007/s00021-013-0159-9.

[16] Baranovskii E.S. 2014. On steady motion of viscoelastic fluid of Oldroyd type. Sbornik: Mathematics. 205(6): 763-776. DOI: 10.1070/SM2014v205n06ABEH004397.

[17] Pileckas K., Sequeira A., Videman J. 2000. Steady flows of viscoelastic fluids in domains with outlets to infinity. Journal of Mathematical Fluid Mechanics, 2: 185-218. DOI: 10.1007/pl00000953.

[18] Talhouk R. 1999. Existence results for steady flow of weakly compressible viscoelastic fluids with a differential constitutive law. Differential and Integral Equations. 12: 741-772.

[19] Artemov M.A., Baranovskii E.S. 2015. Mixed boundary-value problems for motion equations of a viscoelastic medium. Electronic Journal of Differential Equations, 2015 (252): 1-9.

[20] Artemov M.A., Baranovskii E.S. 2015. Initial boundary value problems for viscoelastic Jeffreys fluids. Applied Mathematical Sciences, 9 (121): 6049-6060. DOI: 10.12988/ams.2015.58535

[21] Temam R. 1979. Navier-Stokes Equations. Theory and Numerical Analysis. Amsterdam-New York-Oxford: North-Holland Publishing Company.

[22] Krasnosel'skii M.A., Zabreiko P.P. 1975. Geometrical methods of nonlinear analysis. Moscow: Nauka.