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BOUNDARY ELEMENT METHOD FOR SHEAR DEFORMABLE PLATE WITH MATERIAL NONLINEARIRTY

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ABSTRACT

In this paper a formulation of boundary element method for shear deformable plate theory with material nonlinearity is presented. The material is assumed to undergo small strains. The von Mises criterion is used to evaluate the plastic zone and elastic perfectly plastic material behaviour is assumed. An initial stress formulation is used to formulate the boundary integral equations. Not only the plastic strain due to bending but also the plastic strains due to membrane are considered. The domain integral due to material nonlinearity is evaluated using a cell discretization technique. A total incremental method is applied instead of an incremental and iterative procedure, to solve the nonlinear boundary integral equations. Numerical examples are presented to demonstrate the validity and the accuracy of the formulation.

Keywords: shear deformable plate, material nonlinearity, total increment method, boundary element method.

INTRODUCTION

Plate structures are widely used in engineering applications, for example aircraft, cars, boiler drums, pressure vessels, building slabs, ships, etc. During its services, the plate may be subjected to variety of loads, such as tension, bending or combined tension and bending. In the case of tension, the plate is considered as a two-dimensional problem and it can be solved using the plane stress theory of elasticity (Timoshenko, 1970). On the other hand, the bending problems are explained by plate bending theory (Reissner, 1947, Timoshenko, 1959) and the combined tension and bending cases can be represented by superposition of plate bending theory and two-dimensional plane stress theory.

There are two widely used plate theories. The first one was developed by Kirchhof (1850)] and is commonly referred to as the classical theory. The other was developed by Reissner (1945), and is known as the shear deformable theory. The classical theory is adequate for analyzing certain applications, however, for problems involving stress concentrations and cracks the theory has been shown not to be in agreement with experimental measurements (Aliabadi, 1998). Unlike the classical theory, the Reissner theory takes into account the shear deformation. In the shear deformable plate theory, the problem is represented in terms of three degrees of freedom, involving generalized displacements (i.e. two rotations and deflection) and generalized tractions (i.e. moments and transverse shear forces).

The analysis of plate bending problems by the boundary element method (BEM) has been reported by many researchers. However, its application to elastoplastic plate bending analysis is very limited. The application of the BEM to elastoplastic analysis of Reissner plates can be found in the works by Karam and Telles (1988), Karam and Telles (1992), Karam and Telles (1998) and Ribeiro and Venturini (1998).

Karam and Telles (1992) presented a formulation of elastoplastic analysis of Reissner plates as an extension of their previous work (1988). Later, they also formulated the boundary integral equations for the same application using an initial stress approach and the cell discretization technique was applied to evaluate the domain integral in which the triangular cells are used (1998). To solve the elastoplastic problem an incremental and iterative procedure was adopted together with von Mises and Tresca yield criteria. The classical plasticity theory was used in which plastic strains are time independent. The plastic strain was considered only due to bending and elastic-perfectly plastic material was considered. An alternative approach for dealing with elastoplastic analysis of Reissner plates by BEM was presented by Ribeiro and Venturini (1998). In their work, an incremental and iterative algorithm based on the initial stiffness method was implemented.

This paper presents the boundary element formulation of elastoplastic analysis of Reissner plates. The formulation follows closely the work by Karam and Telles (1998); however not only the plastic strain due to bending but also the plastic strain due to membrane are considered. The total incremental technique is applied in dealing with nonlinear system of equation. The cell discretization method using 9-nodes quadrilateral cell is employed to evaluate the domain integrals appearing in the formulation. Elastic-perfectly plastic material is considered. Throughout this paper, the cartesian tensor notation is used, with Greek indices varying from 1 to 2 and the Latin indices varying from 1 to 3.

GOVERNING EQUATION

In order to define a general formulation for material nonlinearity of plate bending, it is considered that plastic strains are only due to bending and membrane, hence total strain rates can be defined as:

$$\dot{\chi}_{\alpha\beta} = \dot{\chi}_{\alpha\beta}{}^{e} + \dot{\chi}_{\alpha\beta}{}^{a}; \tag{1}$$

$$\dot{\varepsilon}_{\alpha\beta} = \dot{\varepsilon}_{\alpha\beta}{}^{e} + \dot{\varepsilon}_{\alpha\beta}{}^{a}; \tag{2}$$

and

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$$\dot{\gamma}_{\alpha 3} = \dot{\gamma}_{\alpha 3}^{\ e} \tag{3}$$

where, $\dot{\chi}_{\alpha\beta}$ are the total bending strain rates, $\dot{\varepsilon}_{\alpha\beta}$ are the total in-plane strain rates, and $\dot{\gamma}_{\alpha 3}$ shear strain rates respectively. The total bending strain rates consist linear parts $\dot{\chi}_{\alpha\beta}^{\ e}$ and nonlinear parts $\dot{\chi}_{\alpha\beta}^{\ a}$. Similarly total in-plane strain rates consist of linear parts and nonlinear parts. The nonlinear parts of equations (1) and (2) are due to plasticity (p) and they can be expressed as

$$\dot{\chi}_{\alpha\beta}^{\ a} = \dot{\chi}_{\alpha\beta}^{\ p} \tag{4}$$

and

$$\dot{\varepsilon}_{\alpha\beta}^{\ a} = \dot{\varepsilon}_{\alpha\beta}^{\ p} \tag{5}$$

On the other hand, the rates of the stress resultants can also be stated as

$$\dot{M}_{\alpha\beta} = \dot{M}_{\alpha\beta}^{\ e} - \dot{M}_{\alpha\beta}^{\ p}; \tag{6}$$

for moment resultants

$$\dot{Q}_{\alpha} = \dot{Q}_{\alpha}^{\ e}; \tag{7}$$

for shear resultants and

$$\dot{N}_{\alpha\beta} = \dot{N}_{\alpha\beta}^{\ e} - \dot{N}_{\alpha\beta}^{\ p} \tag{8}$$

for membrane stress resultants. The stress-displacement relationships can be presented as

$$\dot{M}_{\alpha\beta} = \frac{1-\upsilon}{2} D\left(\dot{w}_{\alpha,\beta} + \dot{w}_{\beta,\alpha} - \frac{2\upsilon}{1-\upsilon}\dot{w}_{\gamma,\gamma}\,\delta_{\alpha\beta}\right) - \dot{M}_{\alpha\beta}^{\ p}; \tag{9}$$

$$\dot{Q}_{\alpha} = C(\dot{w}_{\alpha} + \dot{w}_{3,\alpha}); \tag{10}$$

and

$$\dot{N}_{\alpha\beta} = \frac{1-\upsilon}{2} B\left(\dot{u}_{\alpha,\beta} + \dot{u}_{\beta,\alpha} - \frac{2\upsilon}{1-\upsilon}\dot{u}_{\gamma,\gamma}\,\delta_{\alpha\beta}\right) - \dot{N}_{\alpha\beta}^{\ p}; \tag{11}$$

where, $D = \dfrac{Eh^3}{1-\upsilon^2}$, $B = \dfrac{Eh}{1-\upsilon^2}$ and $C = \dfrac{Ekh}{2(1+\upsilon)}$.

The equilibrium equation can be expressed as:

$$\dot{M}_{\alpha\beta} - \dot{Q}_{\alpha} = 0; \tag{12}$$

$$\dot{Q}_{\alpha,\alpha} - \dot{q}_{3} = 0; \tag{13}$$

and

$$\dot{N}_{\alpha\beta,\beta} = 0 \tag{14}$$

DISPLACEMENT AND STRESS EQUATION

Applications BEM in solid mechanics are based on the Somigliana's identities. Somigliana's identity for displacements in elastoplastics shear deformable plate bending problems states that the rate of the displacements (two rotations and one deflection) at any points *X'* [$\dot{w}_i(X')$] that belong to domain (*X'Є V*) to the boundary values of displacement rates [$\dot{w}_j(x)$] and traction rates [$\dot{p}_j(x)$] can be expressed as (Karam, 1998):

$$\dot{w}_i(X') = \int_S W_{ij}(X',x)\dot{p}_j(x)dS - \int_S P_{ij}(X',x)w_j(x)dS + \int_V W_{i3}(X',X)\dot{q}_3(X)dV + \int_V \chi_{\alpha\beta i}(X',X)\dot{M}_{\alpha\beta}^{\ p}(X)dV \tag{15}$$

for rotation and deflections

$$\dot{u}_\alpha(X') = \int_S U_{\alpha\beta}(X',x)\dot{t}_\alpha(x)dS - \int_S T_{\alpha\beta}(X',x)\dot{u}_j(x)dS + \int_V \varepsilon_{\theta\alpha\beta}(X',X)\dot{N}_{\alpha\beta}^{\ p}(X)dV \tag{16}$$

for in-plane displacement. Where, $W_{ij}(X',x)$, $P_{ij}(X',x)$, $\chi_{ij}(X',X)$, $U_{ij}(X',x)$, $T_{ij}(X',x)$, and $\varepsilon_{ij}(X',X)$ are called fundamental solutions representing a displacement, a traction and strain in the *j* direction at point *X* due to a unit point force in the *i* direction at point *X'*and can be found in Supriyono (2007).

Equation (15) and (16) are valid for any source points within domain (*X'Є V*), in order to find solutions on the boundary points, it is necessary to consider the limiting process as *X'→ x'Є S*. After limiting process, boundary displacement integral equations can be expressed as

$$C_{ij}\dot{w}_i(x') = \int_S W_{ij}(x',x)\dot{p}_j(x)dS - \int_S P_{ij}(x',x)w_j(x)dS + \int_V W_{i3}(x',X)\dot{q}_3(X)dV + \int_V \chi_{\alpha\beta i}(x',X)\dot{M}_{\alpha\beta}^{\ p}(X)dV \tag{17}$$

and

$$C_{\alpha\beta}\dot{u}_\alpha(x') = \int_S U_{\alpha\beta}(x',x)\dot{t}_\alpha(x)dS - \int_S T_{\alpha\beta}(x',x)\dot{u}_j(x)dS + \int_V \varepsilon_{\theta\alpha\beta}(x',X)\dot{N}_{\alpha\beta}^{\ p}(X)dV \tag{18}$$

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where, $C_{ij}(x')$ are free term that are $C_{ij}(x')=\delta_{ij}(x')+\alpha_{ij}(x')$, for smooth boundary the free term is 0.5.

The Somigliana's identity for stresses can be expressed respectively as:

$$\dot{M}_{\alpha\beta}(X') = \int_S W_{\alpha\beta k}(X',x)\dot{p}_k(x)dS - \int_S P_{\alpha\beta k}(X',x)\dot{w}_k(x)dS + \int_V W_{\alpha\beta k}(X',X)\dot{q}_3(X)dV + \int_V \chi_{\alpha\beta\gamma\theta}(X',X)\dot{M}^{pl}_{\gamma\theta}(X)dV - \frac{1}{8}\left[2(1+\upsilon)M_{\alpha\beta}(1-3\upsilon)\dot{M}^{pl}_{\theta\theta}\delta_{\alpha\beta}\right] \tag{19}$$

for moment resultants,

$$Q_\beta(X') = \int_S W_{\alpha\beta k}(X',x)p_k(x)dS - \int_S P_{\alpha\beta k}(X',x)w_k(x)dS + \int_V W_{3\beta k}(X',V)q_3(X)dV + \int_V \chi_{3\beta\gamma\theta}(X',X)\dot{M}^{pl}_{\gamma\theta}(X)dV \tag{20}$$

for shear resultants, and

$$\dot{N}_{\alpha\beta}(X') = \int_S U_{\alpha\beta\gamma}(X',x)t_\gamma(x)dS - \int_S T_{\alpha\beta\gamma}(X',x)u_\gamma(x)dS + \int_V \varepsilon_{\alpha\beta\gamma\theta}(X',X)\dot{N}^{p}_{\gamma\theta}(X)dV - \frac{1}{8}\left[2(1+\upsilon)N_{\alpha\beta}(1-3\upsilon)N^{p}_{\theta\theta}\delta_{\alpha\beta}\right] \tag{21}$$

for membrane resultants.

DISCRETIZATION AND SYSTEM OF EQUATION

In order to solve equation (17), (18), (19), (20) and (21), a numerical method is implemented. The boundary S is discretized using quadratic isoparametric elements. The domain V is divided into number of cells of 9 nodes quadrilateral cell (as shown in Figure-1).

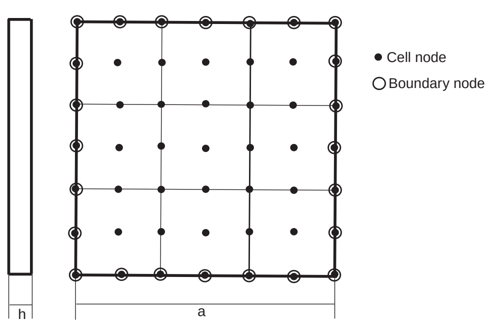

Figure-1. Discretization.

In this formulation, boundary parameter x_j, the unknown boundary values of displacements w_j and tractions p_j are approximated using interpolation function, in following manner:

$$x_j = \sum_{\alpha=1}^{3} N_\alpha(\xi)x_j^\alpha$$
$$\dot{w}_j = \sum_{\alpha=1}^{3} N_\alpha(\xi)\dot{w}_j^\alpha$$
$$\dot{p}_j = \sum_{\alpha=1}^{3} N_\alpha(\xi)\dot{p}_j^\alpha \tag{22}$$

The shape functions N_α are defined as

$$N_1 = \frac{1}{2}\xi(\xi-1)$$
$$N_2 = (1-\xi)(1+\xi)$$
$$N_3 = \frac{1}{2}\xi(\xi+1) \tag{23}$$

After discretization and point collocation on the boundary as well as in the domain, the equations (17) and (18) can be written in the matrix form as

$$\begin{bmatrix} H^w & 0 \\ 0 & H^u \end{bmatrix}\begin{Bmatrix} \dot{w} \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} G^w & 0 \\ 0 & G^u \end{bmatrix}\begin{Bmatrix} \dot{p} \\ \dot{t} \end{Bmatrix} + \begin{Bmatrix} \dot{b} \\ 0 \end{Bmatrix} + \begin{bmatrix} T^w & 0 \\ 0 & T^u \end{bmatrix}\begin{Bmatrix} \dot{M}^p \\ \dot{N}^p \end{Bmatrix} \tag{24}$$

where [H] and [G] are the well-known boundary element influence matrices, [T] is the influence matrix due to plasticity. The superscript w and u show the plate and the in-plane mode respectively {.w}, {u}, {.p} and {t} are the displacement and the traction rate vectors on the boundary. {b} is the load rate vectors on the domain and {M^p} and {N^p} are the nonlinear term due to plasticity. After imposing boundary condition, equations (24) can be written as

$$[A]\{\dot{x}\} = \{f\} + \begin{bmatrix} T^w & 0 \\ 0 & T^u \end{bmatrix}\begin{Bmatrix} \dot{M}^p \\ \dot{N}^p \end{Bmatrix} \tag{25}$$

where, [A] is the system matrix, {x} is the unknown vector and {f } is the vector of prescribed boundary values. Similarly, the stress integral equations of equations (19), (20) and (4) can be presented in matrix form as

$$\begin{bmatrix} \dot{M} \\ \dot{Q} \\ \dot{N} \end{bmatrix} = \begin{bmatrix} G^{w_\alpha} & 0 \\ G^{w_3} & 0 \\ 0 & G^u \end{bmatrix}\begin{Bmatrix} \dot{p} \\ \dot{t} \end{Bmatrix} - \begin{bmatrix} H^{w_\alpha} & 0 \\ H^{w_3} & 0 \\ 0 & H^u \end{bmatrix}\begin{Bmatrix} \dot{w} \\ \dot{u} \end{Bmatrix} + \begin{Bmatrix} b^\alpha \\ b^3 \\ 0 \end{Bmatrix} + \begin{bmatrix} T^{w\alpha} + E^{w\alpha} & 0 \\ T^3 & 0 \\ 0 & T^u \end{bmatrix}\begin{Bmatrix} \dot{M}^p \\ \dot{N}^p \end{Bmatrix} \tag{26}$$

SOLUTION ALGORITHM

The total incremental method solves the nonlinear system of equations of equation (25) based on the incremental procedure. It has an algorithm as:

1. Solve the equation (25), assume that the nonlinear terms (M^p, N^p $=0$) are equal to zero for the first load increment. It means that the linear system equations

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are solved. For the $(k+1)$th load increment it is assumed that $(M^p)_{(k+1)\text{th}}=(M^p)_{k\text{th}}$

2. Solve equation (26) based on the boundary values obtained from number 1. The same case as number 1 is implemented for the nonlinear term.
3. Evaluate of the plastic zone based on the stress obtained from the number 2. In this stage the von Mises criterion is used.
4. If the plasticity has taken place then, obtain the nonlinear term otherwise go to the number 5. The clear explanation of the determination of the plastics term can be found in the work by Supriyono (2007).
5. If the load is less than the final load then go to number 1 and repeat until the load is equal final load

NUMERICAL EXAMPLE

In order to show the validity and the accuracy of the BEM formulation and total increment method in dealing with nonlinear term, an example is presented.

A plane stress problem of a rectangular plate with center hole as shown in Figure-2 is subjected to tension load (σ=50 MPa) as well as uniformly tranverse load (q=0.184 MPa). The tension load is acting on the two side of 20 mm length, whereas the uniformly tranverse load is acting on the whole surface of the plate. The material has properties of E=70 Gpa, ν=0.2 and σ_Y=243 Mpa.

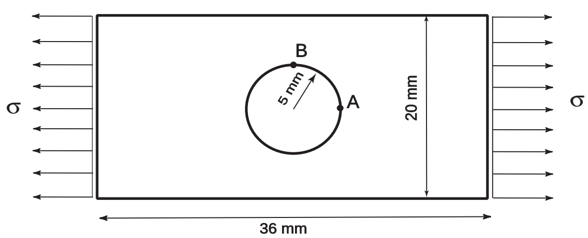

Figure-2. Hole-Plate with tension of σ and surface presure of q.

BEM model (see Figure-3) which has 72 elements on the boundary with 64 domain cells and found to give converged solutions is used in this example.

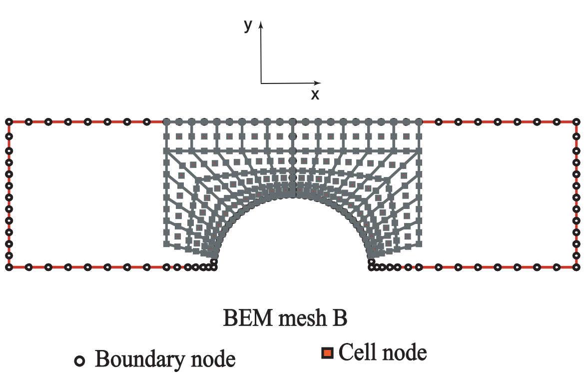

Figure-3. BEM model.

Figure-4 shows the deflections at point B during loading which are compared to the FEM results. The FEM results are obtained using ANSYS with 763 domain cells of plastic 4-node 43 shell element which is also found to give converged results. The nondimensional parameters are defined as in the figure. It can be seen from the figure that BEM and FEM results are in good agreement.

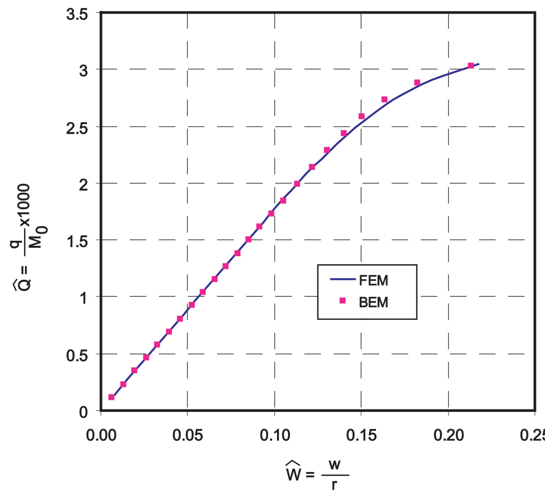

Figure-4. Deflection at point B of the example plate with bending load q and tension load.

Figure-5 presents the displacement contour in the x−direction (Figures a and b) and the deflection contour (Figures c and d) of the example plate in millimeter. It can be seen that the contours look similar, however FEM deflection contour has slightly bigger region with maximum deflection compared to the BEM. The values for both displacement and deflection differ sligthy, but are considered acceptable as the two methods are based on numerical solution.

Plastic zone (von Mises stress) at the final load is presented in Figure-6 for both FEM and BEM analysis in MPa. It can be seen that the contours as well as the values look slightly different; however they are considered acceptable as the two methods are based on numerical solution.

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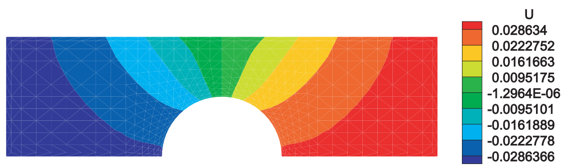
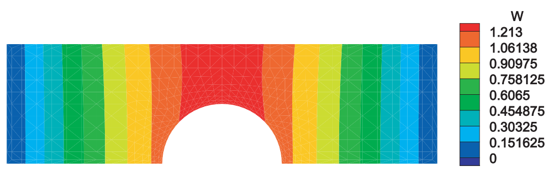
(a) FEM model u contour



(b) BEM model u contour

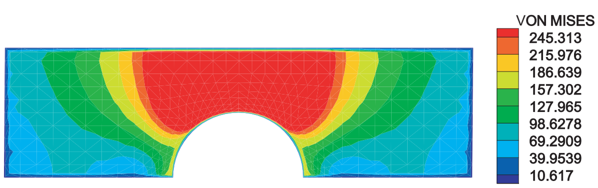
(c) FEM model deflection contour



(d) BEM model deflection contour

Figure-5. Displacement in x−direction and deflection contour for both FEM and BEM analysis.

(a) FEM von Mises stress contour



(b) BEM von Mises stress contour

Figure-6. Plastic zone at the final load for both FEM and BEM analysis.

CONCLUSIONS

The application of BEM to material nonlinearity for shear deformable plate bending analysis was presented and the total incremental method was implemented to solve the nonlinear system of equation, .from the results obtained it can be concluded that:

1. The BEM formulations presented in this paper have good agreement with FEM and it has been shown that BEM model simpler compared to the FEM model.
2. The total incremental method was shown to be an efficient approach for this problem as repeated solution of system of equations is not required and the nonlinear terms are updated by back substitution.

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