



DETECTING THE SMOOTHNESS OF NUMERICAL SOLUTIONS TO THE EULER EQUATIONS OF GAS DYNAMICS

Sudi Mungkasi¹, F. A. R. Sambada² and I. Gusti Ketut Puja²

¹Department of Mathematics, Faculty of Science and Technology, Sanata Dharma University,
Mrican, Tromol Pos 29, Yogyakarta, Indonesia

²Department of Mechanical Engineering, Faculty of Science and Technology, Sanata Dharma University,
Mrican, Tromol Pos 29, Yogyakarta, Indonesia
E-Mail: sudi@usd.ac.id

ABSTRACT

Analytical solutions to the Euler equations of gas dynamics are not available for general cases. The Euler equations are three simultaneous partial differential equations representing the conservation of mass, momentum, and energy of the gas. In this paper, the Euler equations are solved using a finite volume numerical method. As the partial differential equations are of hyperbolic type, their solutions admit discontinuity. The finite volume solution is generally accurate at smooth regions and inaccurate at rough regions. Knowing which regions of the solutions where they are smooth or rough is the goal of this paper. To achieve our goal, here we propose the weak local residual of the entropy equation as a smoothness indicator for numerical solutions to the Euler equations.

Keywords: Euler equations, finite volume method, gas dynamics, smoothness indicator, weak local residual.

INTRODUCTION

The Euler equations have been applied to model gas dynamics [1-4]. They are also applicable in two-phase flow problems consisting of vapour and liquid phases [7]. The solution to the Euler equations can be either continuous or discontinuous. Unfortunately the general analytical solution to the Euler equations is not available so far.

An attempt to get the solution is by using a numerical method, such as a finite volume method which is implemented in this paper. As an approximate solution, the finite volume numerical solution has errors. The finite volume numerical solution is usually accurate for smooth regions. In practice, researchers may need to determine which areas are smooth and rough, so that they can fix the inaccurate parts to be more accurate.

In this paper, we propose to use the weak local residual of the entropy equation of the Euler equations. If the solution is exact, then the residual value is zero. If there are errors, the residual value is non zero. The larger the error implies the larger values of residuals. Weak local residual has also been used in solving the shallow water equations [5-6] and other conservation laws [1].

The next sections contain the main parts of this paper. First, we recall the mathematical model of the Euler equations of gas dynamics. Then we shall explain the numerical method to solve the Euler equations as well as to compute the weak local residual of the entropy equation. It will be followed by the presentation and description of our numerical results. Finally, some concluding remarks will be drawn.

MATHEMATICAL MODEL

The mathematical model for gas dynamics is given by the Euler equations. The Euler equations for mass, momentum, and energy of ideal gases are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad (1)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} = 0 \quad (2)$$

$$\frac{\partial(E)}{\partial t} + \frac{\partial(u(E + p))}{\partial x} = 0 \quad (3)$$

Here, ρ represents the density, u the velocity, p the pressure, E the total energy of the gas. We note the following relations for gas dynamics

$$E = \rho u^2 / 2 + \rho e \quad (4)$$

$$p = \rho e(\gamma - 1) \quad (5)$$

where $\gamma = 1.4$ and e is the specific internal energy. In the vector form, the Euler equations can be written as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = 0 \quad (6)$$

where \mathbf{q} is the vector of conserved quantities and $\mathbf{f}(\mathbf{q})$ is the corresponding vector of flux function.

The entropy relation for the Euler equations is

$$\frac{\partial \eta(\mathbf{q})}{\partial t} + \frac{\partial \psi(\mathbf{q})}{\partial x} \leq 0 \quad (7)$$

which is an equation for smooth solutions and inequality for non-smooth solutions [8]. Here, $\mathbf{q} = [\rho \quad \rho u \quad E]^T$ is the vector of conserved quantities; and



$$\eta(\mathbf{q}) = -\rho \log(\rho e / \rho^\gamma) \quad (8)$$

$$\psi(\mathbf{q}) = -u\eta(\mathbf{q}) \quad (9)$$

are the entropy function and entropy flux respectively. Again with this entropy pair, the entropy relation becomes strict inequality if the solution to the Euler equations is discontinuous. This entropy equation and entropy inequality are understood in the weak sense.

As studied by Hudson [2], the Jacobian matrix of the Euler equations for ideal gases is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & \gamma-1 \\ u(\frac{1}{2}(\gamma-1)u^2 - H) & H - (\gamma-1)u^2 & pu \end{bmatrix} \quad (10)$$

where the total specific enthalpy is

$$H = \frac{E+p}{\rho} = \frac{a^2}{\gamma-1} + \frac{1}{2}u^2 \quad (11)$$

and the sound speed is

$$a = \sqrt{\frac{p}{\rho}} = \sqrt{\frac{\gamma}{\rho}(\gamma-1)(E - \frac{1}{2}\rho u^2)} \quad (12)$$

As calculated by Hudson [2], the eigenvalues of \mathbf{A} are

$$\lambda_1 = u - a, \quad \lambda_2 = u, \quad \text{and} \quad \lambda_3 = u + a \quad (13)$$

corresponding to three different eigenvectors.

NUMERICAL METHOD

For scalar conservation laws

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0, \quad (14)$$

Constantin and Kurganov [1] used a weak local residual as a smoothness indicator for its numerical solutions. Let Δx be the cell width of uniformly discretized space domain, and Δt be a given time step such that the numerical method is stable. The formulation of the weak local residual is

$$R_{j+1/2}^{n-1/2} = \frac{\Delta x}{2} [q_j^n - q_j^{n-1} + q_{j+1}^n - q_{j+1}^{n-1}] + \frac{\Delta t}{2} [f(q_{j+1}^{n-1}) - f(q_j^{n-1}) + f(q_{j+1}^n) - f(q_j^n)] \quad (15)$$

Implementing this formula to the entropy relation, we obtain

$$R_{j+1/2}^{n-1/2} = \frac{\Delta x}{2} [\eta(\mathbf{q}_j^n) - \eta(\mathbf{q}_j^{n-1}) + \eta(\mathbf{q}_{j+1}^n) - \eta(\mathbf{q}_{j+1}^{n-1})] + \frac{\Delta t}{2} [\psi(\mathbf{q}_{j+1}^{n-1}) - \psi(\mathbf{q}_j^{n-1}) + \psi(\mathbf{q}_{j+1}^n) - \psi(\mathbf{q}_j^n)] \quad (16)$$

As we mentioned, we use a finite volume numerical method to solve the Euler equations. In a semi-discrete form, the finite volume method is

$$\frac{d}{dt} \mathbf{Q}_j(t) = -\frac{1}{\Delta x} (\mathbf{F}_{j+\frac{1}{2}}(t) - \mathbf{F}_{j-\frac{1}{2}}(t)) \quad (17)$$

where \mathbf{Q} is the vector of approximate conserved quantities, \mathbf{F} is the vector of approximate fluxes and \mathbf{S} is the vector of the source term discretization. This scheme is called semi-discrete because we have discretized the Euler equations with respect to space, but the time variable is still continuous.

The next step is to integrate the semi-discrete form (17) with respect to time. We can use any standard method of Ordinary Differential Equations (ODEs) solver. However, because we have used a first order method in space, it is better to use a first order method in time. This is because we will never get a finite volume method of order higher than one, even if we use higher order method in time. In this paper we implement the first order Runge-Kutta method to integrate the semi-discrete form (17) with respect to time. Therefore, the fully discrete finite volume method is

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x} (\mathbf{F}_{j+\frac{1}{2}}^n - \mathbf{F}_{j-\frac{1}{2}}^n).$$

We use the Lax-Friedrichs numerical flux function. We refer to LeVeque [4] and Puppo and Semplice [8] for the formulation of this flux function. Therefore, for non-boundary fluxes we obtain

$$\mathbf{F}_{j+1/2} = \frac{1}{2} (\mathbf{F}_j + \mathbf{F}_{j+1}) - \frac{a_{j+1/2}}{2} (\mathbf{Q}_{j+1} - \mathbf{Q}_j).$$

Here we have dropped the superscript n for writing simplicity, and note that

$$a_{j+1/2} = \max\{a_j, a_{j+1}\}.$$

The time step Δt is taken by considering the CFL (Courant-Friedrichs-Lewy) number, which is formulated as

$$\Delta t = \frac{\text{CFL} \cdot \Delta x / 2}{\max\{a\}}$$

where a is the sound speed and the maximum of a is taken for the whole space domain.



NUMERICAL RESULTS

We consider the Sod's shock tube problem [9]. This problem has also been used as a benchmark problem by a number of authors, such as Hudson [2] and Laney [3] to test the performance of their numerical methods.

Suppose that we have a gas tube separated by a diaphragm so that there are two states of gases. The gas on the left of the diaphragm has $\rho=1, u=0$ and $p=1$. On the other side, the gas on the right has $\rho=0.125, u=0$ and $p=0.1$. Here we have assumed to use SI units. The position of the diaphragm is at $x=0$. Using the aforementioned finite volume method, numerical results are recorded for time $t=0.25$.

Discretizing the space domain $[-0.5, 0.5]$ into

1000 cells and using CFL number 0.3, the L^1 error of the density is 0.0078. This error is less than 1% which is very good quantitatively. Note that the the L^1 error formula that we use is

$$E = \frac{1}{N} \sum_{j=1}^N |q_j - Q_j|$$

where N is the number of cells, q_j is the exact value of the quantity at j th cell, and Q_j is the numerical value of the quantity at j th cell.

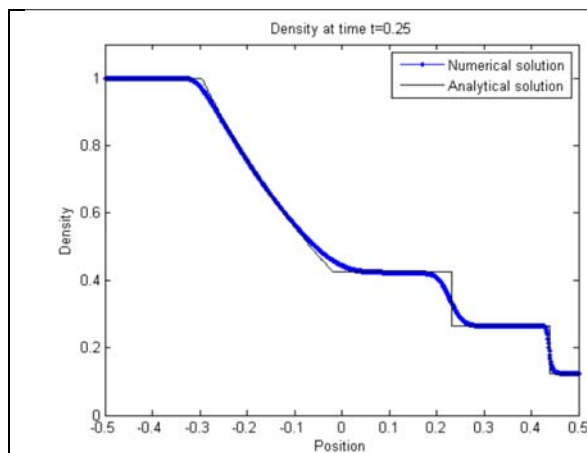


Figure-1. Density results of the shock tube problem with 1000 cells at time $t=0.25$.

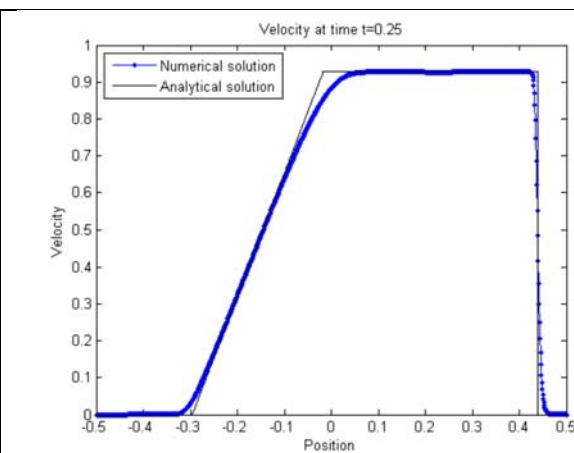


Figure-2. Velocity results of the shock tube problem with 1000 cells at time $t=0.25$.

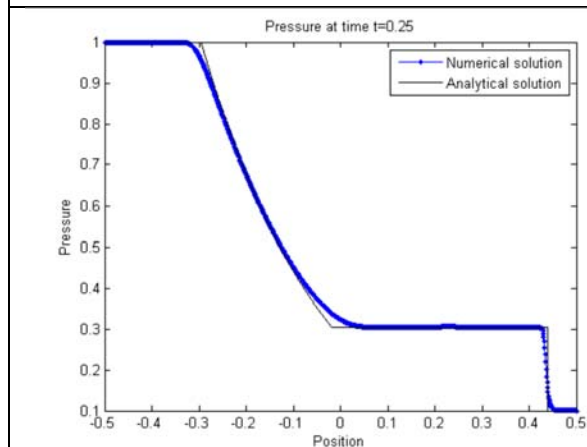


Figure-3. Pressure results of the shock tube problem with 1000 cells at time $t=0.25$.

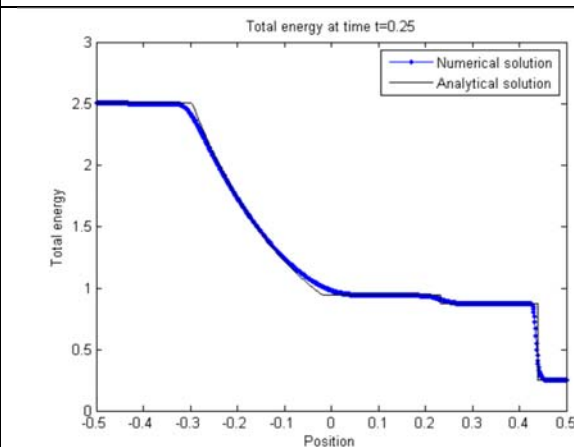


Figure-4. Total energy results of the shock tube problem with 1000 cells at time $t=0.25$.

The exact and the numerical results are shown in Figures. Figure-1 shows the density, Figure-2 the velocity, Figure-3 the pressure, Figure-4 the total energy, Figure-5 the internal energy, and Figure-6 the weak local residual of the entropy equation.

Based on the error and pictorial results, the finite volume method produce quite good solution. However, we

notice that the finite volume numerical solution is diffusive at around corners and discontinuities. This is well-captured by our proposed smoothness indicator, the weak local residual of the entropy equation. The smoothness indicator produces large values at around corners and discontinuities, where the numerical solution is diffusive, that is, inaccurate. These are the places



needing special treatment in order to get more accurate numerical solutions. However, this special treatment is beyond our goal in this paper.

In the solution of this shock tube problem, we obtain three propagating waves. The first is a rarefaction fan moving and expanding to the left. The second is a contact discontinuity propagating to the right. This discontinuity is the position where the two gases remain

separated. The third is the shock discontinuity. This is consistent with the results of Laney [3]; Sod [9] and others (see the References).

In summary of our results, the weak local residual has the correct behaviour as expected. This smoothness indicator is a good candidate to be implemented in adaptive numerical methods for the Euler equations, which we will consider for future research work.

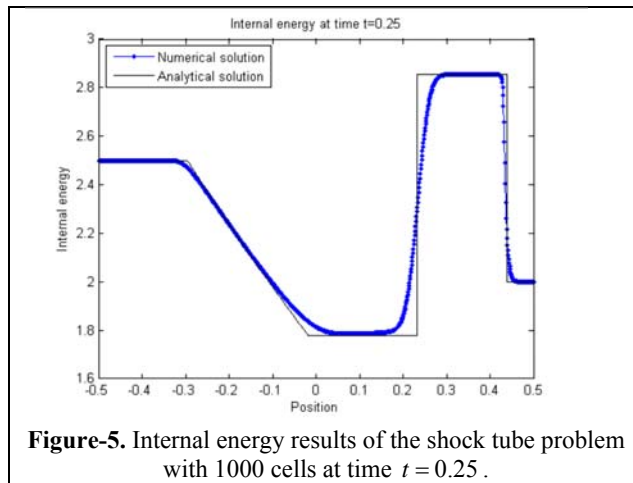


Figure-5. Internal energy results of the shock tube problem with 1000 cells at time $t = 0.25$.

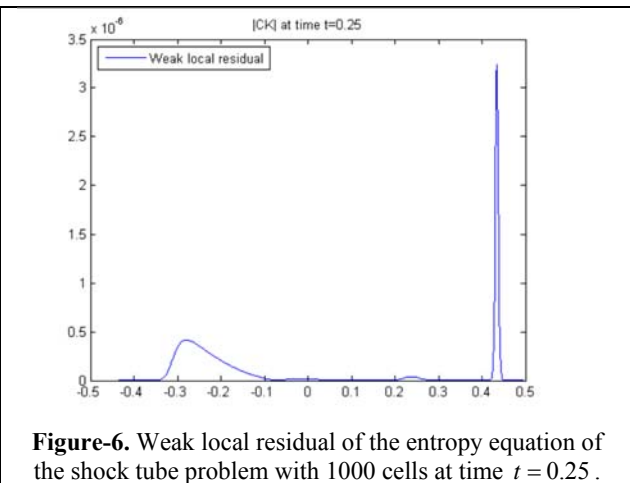


Figure-6. Weak local residual of the entropy equation of the shock tube problem with 1000 cells at time $t = 0.25$.

CONCLUSIONS

A smoothness indicator for numerical solutions to the Euler equations has been proposed. The smoothness indicator is the weak local residual of the entropy equation relating to the Euler equations for ideal gases. Numerical results show that our proposed method can accurately detect the places where numerical solutions are smooth and rough. It suggests that this smoothness indicator is a good candidate to be implemented in adaptive numerical methods for the Euler equations.

ACKNOWLEDGEMENT

This work was financially supported by a research grant of The Indonesian Directorate General of Higher Education (DIKTI).

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