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NUMERICAL SOLUTION OF THE FORCED KORTEWEG-DE VRIES (FKDV) EQUATION

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ABSTRACT

In this paper, the application of the method of lines (MOL) to the FKdV equation is presented. The MOL is a general technique for solving partial differential equations by typically using finite-difference relationships for the spatial derivatives and ordinary differential equations (ODEs) for the time derivative. The MOL approach of the FKdV equation led to a system of ODEs. Solution of the system of ODEs was obtained by applying fourth order Runge Kutta (RK4) method. In order to show the accuracy of the presented method, the numerical solutions obtained were compared with progressive wave solution.

Keywords: FKdV equation, the method of lines, system of differential equation, runge kutta.

INTRODUCTION

It seems that Patoine and Warn (1982) were the first used the forced Korteweg-de Vries (FKdV) equation as a physical model equation in the context of the interaction of long, quasi-stationary, baroclinic waves with topography, given by

$$A_t + \alpha A_x + \beta A A_x + \gamma A_{xxx} = f'(x). \tag{1}$$

where $A(x,t)$ represent the free surface displacement from its undisturbed position at time t, α represent the long-wave speed, β represent the strenght of the nonlinearities, $\gamma < 0$ and $f'(x)$ represent the forcing terms. However, Akylas (1984) was the first systematically derived the FKdV equation from the model of long nonlinear water waves forced by a moving pressure as shown below:

$$6\eta_t + \eta_{xxx} + 9\eta\eta_x - 6(F-1)\eta_x = -3p_x. \tag{2}$$

where $\eta(x,t)$ represent the first order elevation of the free surface of the fluid from its equilibrium point t,. Equation (2) is strictly only valid for small disturbances and when the Fraud number F is close to unity and p is forcing term.

The one-dimensional stationary FKdV equation was derived by Shen (1995) as follows:

$$\lambda U_t - \frac{3}{2}UU_x - \frac{1}{6}U_{xxx} = \frac{1}{2}h_x, \tag{3}$$

where $U(x,t)$ represent free surface elevation for the long nonlinear water waves flowing over long bump, $\lambda > 0$ and h represent the forcing term. Shen (1995) proved that the existence of positive solitary wave solutions to the equation (3) with boundary conditions $u(\pm\infty) = u'(\pm\infty) = 0$. recently, the analytical solution of (3) which is a certain form of forcing term has been solved by Zhao and Guo (2009) by using Hirota Direct method.

In literature, weakly nonlinear wave propagation in a prestressed fluid-filled stenosed elastic tube filled with an inviscid fluid has been studied by applying the reductive perturbation method and in the long wave approximation (Tay, 2007). By using the stretched coordinate of initial-value type and expanding the field quantity into the asymptotic series of small parameter, ε, where ε is a small parameter, the governing equations are reduced to the FKdV equation with the variable coefficients, that is,

$$u_\tau + \mu_1 u u_\xi + \mu_3 u_{\xi\xi\xi} + \mu_4(\tau) u_\xi = \mu(\tau) \tag{4}$$

where ξ is a spatial variable, τ is a temporal variable, $\mu_1, \mu_3, \mu_4(\tau), \mu(\tau)$, are the coefficients of nonlinearity, dispersion, variable coefficient and forcing term respectively. The presence of forcing term $\mu(\tau)$, and variable coefficient term, $\mu_4(\tau)u_\xi$ show the presence of stenosis. The coefficients of $\mu_1, \mu_3, \mu_4(\tau)$ and $\mu(\tau)$ are defined by (Tay, 2007) as

$$\mu_1 = \frac{5}{2\lambda_\theta} + \frac{\beta_2}{\beta_1}, \quad \mu_3 = \frac{m}{4\lambda_z} + \frac{\lambda_\theta^2}{16} - \frac{\beta_0}{2\beta_1},$$
$$\mu_4(\tau) = \frac{\lambda_\theta \gamma_2}{\beta_1} G(\tau) - \left[\frac{\beta_2}{\beta_1} + \frac{1}{2\lambda_\theta}\right] g(\tau), \quad \mu(\tau) = \frac{1}{2} g'(\tau) - \frac{\lambda_\theta \gamma_1}{2\beta_1} G'(\tau). \tag{5}$$

where

$$\gamma_0 = \frac{1}{\lambda_\theta \lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2}\right) H(\lambda_\theta, \lambda_z),$$

$$\gamma_1 = \frac{1}{\lambda_\theta \lambda_z}\left[\left(1 + \frac{3}{\lambda_\theta^4 \lambda_z^2}\right) + 2\alpha\left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2}\right)^2\right] H(\lambda_\theta, \lambda_z),$$

$$\gamma_2 = \frac{1}{2\lambda_\theta \lambda_z}\left[-\frac{12}{\lambda_\theta^5 \lambda_z^2} + 6\alpha\left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2}\right)\left(1 + \frac{3}{\lambda_\theta^4 \lambda_z^2}\right) + 4\alpha^2\left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2}\right)^3\right] H(\lambda_\theta, \lambda_z),$$

$$\beta_0 = \frac{1}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2 \lambda_z^3}\right) F(\lambda_\theta, \lambda_z),\ \beta_1 = \gamma_1 - \frac{\gamma_0}{\lambda_\theta},\ \ \beta_2 = \gamma_2 - \frac{\beta_1}{\lambda_\theta}. \tag{6}$$

Given that $H(\lambda_\theta, \lambda_z) = \exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2 \lambda_z^2} - 3\right)\right]$, $\alpha = 1.948,\ \lambda_z = \lambda_\theta = 1.6,\ m = 0.1,\ G(\tau) = 0$ and $g(\tau) = \operatorname{sech}(0.01\tau)$. Here α refers to material constant, λ_θ is the initial circumferential stretch ratio, λ_z is the initial axial stretch ratio and m is mass of artery.

The application of the MOL to the FKdV equation (4) will be presented in this paper. The MOL approach of the FKdV equation led to a system of ODEs. Solution of the system was obtained by applying the RK4 method. The solution of the FKdV that is obtained using the MOL with progressive wave solution conducted by (Tay, 2007) will be compared in terms of its maximum absolute error at certain time τ.

THE METHOD OF LINES

The MOL is a powerful method used to solve partial differential equations (PDEs). It involves making an approximation to the spatial derivatives and reducing the problem into a system of ODEs (Hall and Watt, 1976; Loeb and Schiesser, 1973; Schiesser, 1994). In addition, this system of ODEs can be solved by using time integrator. The most important advantage of the MOL approach is that it has not only the simplicity of the explicit methods (Dehgan, 2006) but also the superiority (stability advantage) of the implicit ones unless a poor numerical method for solution of ODEs is employed. It is possible to achieve higher-order approximations in the discretization of spatial derivatives without significant increases in the computational complexity. This method has wide applicability to physical and chemical systems modeled by PDEs. This method has the wide applicability to physical and chemical systems modeled by PDEs such as delay differential equations (Koto, 2004), two-dimensional sine-Gordon equation (Bratsos, 2007), the Nwogu one-dimensional extended Boussinesq equation (Hamdi *et al*, 2005), the fourth-order Boussinesq equation, the fifth-order Kaup–Kupershmidt equation and an extended Fifth-Order Korteweg-de Vries (KdV5) equation (Saucez *et al*, 2004).

In this paper, the spatial derivatives are firstly discretized using central finite difference formulae as follows:

$$\left.\begin{aligned} u &\approx \frac{u_{j+1} + u_j + u_{j-1}}{3}, \\ u_\xi &\approx \frac{u_{j+1} - u_{j-1}}{2\Delta\xi}, \\ u_{\xi\xi\xi} &\approx \frac{u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}}{2(\Delta\xi)^3}, \end{aligned}\right\} \tag{7}$$

where ξ is the spatial variable, τ is the temporal variable, j is the index denoting the spatial position along ξ-axis and $\Delta\xi$ is the step size along the axis. The ξ-interval is divided into M points with $j = 1, 2, ..., M-1, M$. Therefore, the MOL approximation of (4) is given by

$$\begin{aligned} \frac{\partial u_j}{\partial \tau} = &-\frac{\mu_1}{6\Delta\xi}(u_{j+1} + u_j + u_{j-1})(u_{j+1} - u_{j-1}) \\ &-\frac{\mu_3}{2(\Delta\xi)^3}(u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}) \\ &-\frac{\mu_4(\tau)}{2\Delta\xi}(u_{j+1} - u_{j-1}) + \mu(\tau) \equiv f(u_j). \end{aligned} \tag{8}$$

Equation (8) is written as an ODE since there is only one independent variable, which is τ. Also, (8) represents a system of M equations of ODEs. The initial condition for (8) after discretization is given by

$$u(\xi_j, \tau = 0) = u_0(\xi_j), \qquad j = 1, 2, ..., M-1, M. \tag{9}$$

For the time integration, the RK4 method is applied. Thus, the numerical solution at time τ_{i+1} is

$$u_{i+1,j} = u_{i,j} + \frac{1}{6}(a_{i,j} + 2b_{i,j} + 2c_{i,j} + d_{i,j}), \tag{10}$$

where

$$\left.\begin{aligned} a_{i,j} &= \Delta\tau\, f\left(u_{i,j}\right), \\ b_{i,j} &= \Delta\tau\, f\left(u_{i,j} + \frac{1}{2}a_{i,j}\right), \\ c_{i,j} &= \Delta\tau\, f\left(u_{i,j} + \frac{1}{2}b_{i,j}\right), \\ d_{i,j} &= \Delta\tau\, f\left(u_{i,j} + c_{i,j}\right). \end{aligned}\right\} \tag{11}$$

Here $\Delta\tau$ is the step size of the temporal coordinate.

PROGRESSIVE WAVE SOLUTION

The progressive wave solution of the FKdV equation (4) given by (Tay, 2007) is

$$U = a\,\mathrm{sech}^2\,\zeta + \frac{1}{2}\left[g(\tau) - \frac{\lambda_\theta \gamma_1}{\beta_1} G(\tau)\right], \tag{12}$$

where a is the amplitude of the solitary wave. The phase function ζ can be expressed as

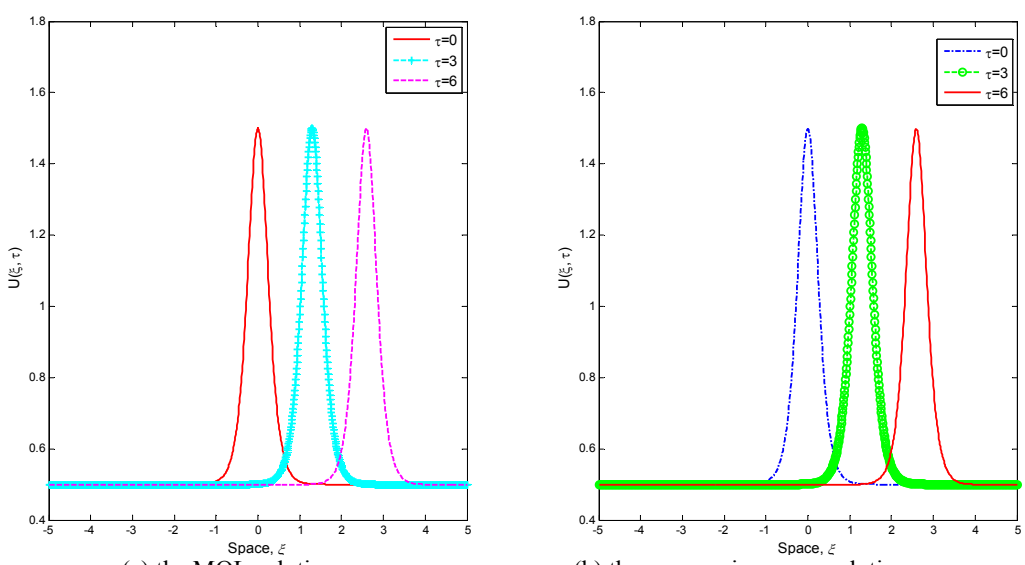
$$\zeta = \left(\frac{\mu_1 a}{12\mu_3}\right)^{\frac{1}{2}} \left\{\xi - \frac{\mu_1 a}{3}\tau - \int_0^\tau \left[\left(\frac{3}{4\lambda_\theta} - \frac{\beta_2}{2\beta_1}\right) g(s) + \frac{\lambda_\theta}{\beta_1}\left(\gamma_2 - \frac{\mu_1\gamma_1}{2}\right) G(s)\right] ds\right\} \tag{13}$$

RESULTS AND DISCUSSIONS

To test ~~the~~ MOL on the FKdV equation (4), we need the initial condition as follows:

$$u(\xi,0) = \mathrm{sech}^2 \sqrt{\frac{\mu_1}{12\mu_3}}\,\xi + 0.5. \tag{14}$$

Figure-1 (a) gives the MOL solution of the FKdV equation (4) with spatial parameters at certain time τ, while Figure-1 (b) represents the progressive wave solution of the FKdV equation (4) with spatial parameters at certain time τ. The solution of the FKdV equation (4) with space ξ shows the solitary waves with the amplitude of one unit propagates to the right as time τ increases. From the observation of Figure-1 (a) and Figure-1 (b), the graph of the solution of the FKdV equation (4) using HPM is exactly the same graph with the progressive wave solution of the FKdV equation (4).



(a) the MOL solution (b) the progressive wave solution

Figure-1. Solution of the FKdV equation versus space ξ for different time τ at $\Delta\xi = 0.01$.

We then computed the absolute error between the progressive wave and MOL solutions for each discretized spatial point at certain time τ in order to calculate the accuracy of the MOL solution and later find the maximum absolute error. The maximum absolute errors between the progressive wave and MOL solutions are calculated based on the formula:

$$L_\infty = \max \left|U_{progressive} - U_{MOL}\right|. \tag{15}$$

Table-1 gives the maximum absolute error between the progressive wave solution and MOL solution.

It shows the maximum absolute errors are in order of 10^{-3}.

Table-1. Maximum absolute error of the FKdV equation for different time τ at $\Delta\tau = \partial\xi^{-3}$.

Time, τ	0	3	6
L_∞	0	4.7508×10^{-3}	7.2510×10^{-3}

CONCLUSIONS

The MOL was employed to solve the FKdV equation was discussed. It involved replacing the spatial derivatives in the PDE with finite-difference approximations and by doing that, the spatial derivatives became are longer stated explicitly in terms of spatial independent variables. This leads to a system of ODEs. The system is then solved by using the RK4 method.

This paper describes the effect of computational effort with respect to the accuracy of results. The MOL solution of the FKdV equation (4) is plotted versus its progressive wave solution. From the observation, it was found that there were no differences for both MOL and progressive wave solutions. The maximum absolute errors between both MOL and progressive wave solutions at certain time τ are computed. Results revealed that the the maximum absolute errors are in the order of 10^{-3} for $\Delta\xi = 0.01$ and $\Delta\tau = 1\times10^{-6}$. Hence, it can be concluded that the FKdV equation can be solved successfully using the MOL.

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