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FEM BASED COLLOCATION METHOD FOR SOLVING EIGHTH ORDER BOUNDARY VALUE PROBLEMS USING B-SPLINES

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ABSTRACT

An easy to implement FEM based collocation method is proposed to solve a special case eighth order boundary value problem. By the proposed method, numerical results canbe obtained not only for the solution but also for derivatives of the solution. Ninth degree B-splines are used as basis functions to approximate the solution. These functions are changed into a set of new functions with the help of boundary conditions. The proposed method with the new set of B-splines gives a stable system of linear equation in the unknown parameters which are used to approximate the solution and its derivatives. To test the efficiency of the method, some numerical examples which are available in literature are solved using the proposed method. The obtained results are in good agreement with the exact solutions.

Keywords: collocation method, eighth order bvp, B-spline approximation, numerical solution.

1. INTRODUCTION

The existence of eighth order boundary value problems has been present in the literature. For one such, when an infinite horizontal layer of fluids is heated from below under the action of rotation, instability sets in. When the instability sets as an over convection, that leads to an eighth order boundary value problem [1]. In this paper a special case eighth order boundary value problem (BVP) of the type

$$y^{(8)} + f(x)y = g(x) \qquad a \le x \le b \; (1)$$
$$y(a) = K_0 y'(a) = K_1 y''(a) = K_2 \quad y'''(a) = K_3$$
$$y(b) = L_0 y'(b) = L_1 y''(b) = L_2 \quad y'''(b) = L_3$$

where $f(x)$ and $g(x)$ are continuous functions on $[a, b]$ and K_0, K_1, K_2, K_3, L_0, L_1, L_2 and L_3 are finite real constants, is considered. The existence and uniqueness of solution for this type problems discussed in the book written byAgarwal [2]. Solving this type of boundary value problems in analytical way is somewhat difficult and thus the necessity of using a numerical method is required to solve this type of problems.

There are several authors who worked on this type of problems using various methods. Ghazala Akram and others [4] solved these type problems using the technique of reproducing kernel space. Varaitional iteration methods are used by [7], [9] and [12]. Galerkin method with different basis functions was applied to solve this type of problems by KasiViswanadham[6] and ShafiqulIslam[8]. Some other numerical techniques were implemented by the authors [5], [11] and [13]. The proposed method involves collocation method with an approximation to the solution as linear combination of ninth degree B-splines. The main goal of this paper is to present a simple and easy to implement numerical method which approximates not only the solution but also its derivatives of an eighth order boundary value problem.

This paper is divided into seven sections. In the next section a brief introduction about B-splines and its nature at grid points has discussed. In section 3, collocation method for solving an eighth order boundary value problem has discussed. The stability of the method is presented in section 4. Numerical examples showing the efficiency of the method is presented in section 6. Finally the last section is dedicated to the conclusions.

2. B-SPLINES

B-splines are the basis functions for the space of spline functions. If we approximate a curve $f(x)$ by a ninth degree spline$\hat{f}(x)$, then $\hat{f}(x)$ can be represented uniquely by a linear combination of B-spline functions. Usage of these functions is vast in many upcoming research areas viz approximation theory, image processing, computer graphics etc.

A ninth degree B-spline $B_i(x)$ with arbitrarily spaced knots t_{i-5}, t_{i-4}, t_{i-3}, ..., t_{i+4}, t_{i+5} given in [10] is

$$B_i(x) = \frac{1}{h^9}\begin{cases} \sum_{r=0}^{10} (-1)^r \binom{10}{r} (t_{i+r-5} - x)_+^9, \text{if} x \in [t_{i-5}, t_{i+5}] \\ 0, \text{if} x \in]t_{i-5}, t_{i+5}[\end{cases} (2)$$

Where
$(t_r - x)_+^9 = (t_r - x)^9$,for$t_r \ge x$
$= 0$,for $t_r \le x$

3. COLLOCATION METHOD

To solve the eighth order special case BVP (1) by the collocation method, first consider a uniform grid of length h with sufficient number of intervals $n \ge 10$ over the domain of the problem. Let it be
$a = t_0 < t_1 < \cdots < t_n = b$

Extending the grid points on both the sides of the domain with the same h such that the grid points t_{-9}, t_{-8}, ..., t_{-1} are on left side of the interval $[a, b]$ and t_{n+1}, t_{n+2}, ..., t_{n+9} are on right side of the interval $[a, b]$. With the extended grid system, the set $\{B_{-4}, B_{-3}, \dots, B_{n+4}\}$forms a basis for the space of all ninth degree spline curves over the interval $[a, b]$ with the grid points t_i's.

The approximation for the required solution $y(x)$ is defined as

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$$y(x) \approx \tilde{y}(x) = \sum_{j=-4}^{n+4} \alpha_j B_j(x) \tag{3}$$

where α_j's are the nodal parameters to be determined.

Consider the approximation $\tilde{y}$ of y at the given boundary conditions. From the zeroth derivative boundary conditions, we get

$$\tilde{y}(t_0) = \sum_{j=-4}^{4} \alpha_j B_j(t_0) = K_0 \tag{4}$$

$$\tilde{y}(t_n) = \sum_{j=n-4}^{n+4} \alpha_j B_j(t_n) = L_0 \tag{5}$$

Eliminating the parameters α_{-4}, α_{n+4} from the above two equations (4) and (5), we get

$$\tilde{y}(x) = w_1(x) + \sum_{j=-3}^{n+3} \alpha_j \phi_j(x)$$

where

$$w_1(x) = \frac{K_0}{B_{-4}(t_0)} B_{-4}(x) + \frac{L_0}{B_{n+4}(t_n)} B_{n+4}(x)$$

$$\phi_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(t_0)}{B_{-4}(t_0)} B_{-4}(x), & j = -3, \dots, 4 \\ B_j(x), & j = 5, 6, \dots, n-5 \\ B_j(x) - \dfrac{B_j(t_n)}{B_{n+4}(t_n)} B_{n+4}(x), & j = n-4, \dots, n+3. \end{cases}$$

From the first derivative boundary conditions, we get

$$\tilde{y}'(t_0) = w_1'(t_0) + \sum_{j=-3}^{4} \alpha_j \phi_j'(t_0) = K_1 \tag{7}$$

$$\tilde{y}'(t_n) = w_1'(t_n) + \sum_{j=n-4}^{n+3} \alpha_j \phi_j'(t_n) = L_1 \tag{8}$$

Eliminating the parameters α_{-3}, α_{n+3} from the above two equations (7) and (8), we get

$$\tilde{y}(x) = w_2(x) + \sum_{j=-2}^{n+2} \alpha_j Q_j(x)$$

where

$$w_2(x) = w_1(x) + \left[\frac{K_1 - w_1'(t_0)}{\phi_{-3}'(t_0)}\right] \phi_{-3}(x) + \left[\frac{L_1 - w_1'(t_n)}{\phi_{n+3}'(t_n)}\right] \phi_{n+3}(x)$$

And

$$Q_j(x) = \begin{cases} \phi_j(x) - \dfrac{\phi_j'(t_0)}{\phi_{-3}'(t_0)} \phi_{-3}(x), & j = -2, \dots, 4 \\ \phi_j(x), & j = 5, 6, \dots, n-5 \\ \phi_j(x) - \dfrac{\phi_j'(t_n)}{\phi_{n+3}'(t_n)} \phi_{n+3}(x), & j = n-4, \dots, n+2. \end{cases}$$

In a similar way from the second derivative boundary conditions, we have

$$\tilde{y}''(t_0) = w_2''(t_0) + \sum_{j=-2}^{4} \alpha_j Q_j''(t_0) = K_2 \tag{9}$$

$$\tilde{y}''(t_n) = w_2''(t_n) + \sum_{j=n-4}^{n+2} \alpha_j Q_j''(t_n) = L_2 \tag{10}$$

After the elimination of parameters α_{-2}, $\alpha_{n+2}$the approximation $\tilde{y}(x)$ changes to

$$\tilde{y}(x) = w_3(x) + \sum_{j=-1}^{n+1} \alpha_j R_j(x)$$

where

$$w_3(x) = w_2(x) + \left[\frac{K_2 - w_2''(t_0)}{Q_{-2}''(t_0)}\right] Q_{-2}(x) + \left[\frac{L_2 - w_2''(t_n)}{Q_{n+2}''(t_n)}\right] Q_{n+2}(x)$$

and

$$R_j(x) = \begin{cases} Q_j(x) - \dfrac{Q_j''(t_0)}{Q_{-2}''(t_0)} Q_{-2}(x), & j = -1, \dots, 4 \\ Q_j(x), & j = 5, 6, \dots, n-5 \\ Q_j(x) - \dfrac{Q_j''(t_n)}{Q_{n+2}''(t_n)} Q_{n+2}(x), & j = n-4, \dots, n+1. \end{cases}$$

At last using third derivative boundary conditions, we get

$$\tilde{y}'''(t_0) = w_3'''(t_0) + \sum_{j=-1}^{4} \alpha_j R_j'''(t_0) = K_3 \tag{11}$$

$$\tilde{y}'''(t_n) = w_3'''(t_n) + \sum_{j=n-4}^{n+1} \alpha_j R_j'''(t_n) = L_3 \tag{12}$$

Eliminating the two parameters α_{-1}, α_{n+1} from the above equation (11) and (12) we get the revised approximation for $\tilde{y}(x)$ as

$$\tilde{y}(x) = w(x) + \sum_{j=0}^{n} \alpha_j \tilde{B}_j(x) \tag{13}$$

where

$$w(x) = w_3(x) + \left[\frac{K_3 - w_3'''(t_0)}{R_{-1}'''(t_0)}\right] R_{-1}(x) + \left[\frac{L_3 - w_3'''(t_n)}{R_{n+1}'''(t_n)}\right] R_{n+1}(x)$$

and

$$\tilde{B}_j(x) = \begin{cases} R_j(x) - \dfrac{R_j'''(t_0)}{R_{-1}'''(t_0)} R_{-1}(x), & j = 0, \dots, 4 \\ R_j(x), & j = 5, 6, \dots, n-5 \\ R_j(x) - \dfrac{R_j'''(t_n)}{R_{n+1}'''(t_n)} R_{n+1}(x), & j = n-4, \dots, n. \end{cases}$$

Applying the collocation method with the revised approximation $\tilde{y}(x)$ to the problem (1), we get

$$\left[w^{(8)}(t_i) + \sum_{j=0}^{n} \alpha_j \tilde{B}_j^{(8)}(t_i)\right] + f(x_i)\left[w(t_i) + \sum_{j=0}^{n} \alpha_j \tilde{B}_j(t_i)\right] = g(t_i) \quad i = 0, 1, \dots, n. \tag{14}$$

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Rewriting the above system of equations in the matrix form, we get

$$\mathbf{K}\alpha = \mathbf{F} \tag{15}$$

where

$$\begin{aligned}
\mathbf{K} &= [a_{ij}]; \quad \mathbf{F} = [b_i]; \\
a_{ij} &= \tilde{B}_j^{(8)}(t_i) + f(t_i)\tilde{B}_j(t_i) \\
b_i &= g(t_i) - w^{(8)}(t_i) - f(t_i)w(t_i) \\
& \quad for\, i, \quad j = 0,1,2,\ldots,n \\
\alpha &= [\alpha_0 \alpha_1 \ \ldots \alpha_n]^T
\end{aligned}$$

4. STABILITY OF THE SYSTEM

The B-spline function and its eighth derivative satisfy the following conditions

$|B_i(t_i)| > |B_i(t_j)| \quad for\, j \neq i$
$|B_i^{(8)}(t_i)| > |B_i^{(8)}(t_j)| \quad for\, j \neq i$
and $B_i^{(k)}(x) = 0 \quad for\, x \in]t_{i-5}, t_{i+5}[$

The function $\tilde{B}_i(x)$ and $\tilde{B}_i^{(8)}(x)$ are linear combination of $B_i(x)$ and $B_i^8(x)$ respectively, for $i \leq 4$ and $i \geq n-4$ and equal to $B_i(x)$ and $B_i^8(x)$for $i = 5,6,\ldots,n-5$.

Hence, it is evident that $|a_{ii}| > |a_{ij}|$ and this leads the dominant property for the matrix **K**. Also $B_i(t_j) = 0$ and $B_i^{(8)}(t_j) = 0$ for $j \leq i-5$and $j \geq i+5$. Therefore, the matrix **K** is a nine band diagonal matrix with dominant property and hence it is nonsingular.

5. SOLUTION AND ITS DERIVATIVES

The unknown nodal parameters α_j's can be obtained by solving the system (15). After obtaining the unknown parameters, the solution and its derivatives of the considered problem are approximated by the following:

$$y^{(k)}(x) \approx \tilde{y}^{(k)}(x) = w^{(k)}(x) + \sum_{j=0}^{n} \alpha_j \tilde{B}_j^{(k)}(x)$$
$k = 0,1,\ldots,8$

6. NUMERICAL EXAMPLES

The proposed method is tested for solving few problems which are available in the literature (with exact solution). Each problem is solved by the proposed method with 10 grid points. The obtained solution and its derivatives (first and second) are observed at randomly selected points in the domain. Mean Absolute Error (MAE) is computed for the solution and its derivatives of each problem.

Example 1: Consider the following eighth order linear boundary value problem

$$y^{(8)} + xy = -(48 + 15x + x^3)e^x \qquad 0 \leq x \leq 1$$
$$y(0) = y(1) = 0 \quad y'(0) = 1 \quad y'(1) = -ey''(0) = 0$$
$$y''(1) = -4ey'''(0) = -3 \quad y'''(1) = -9e$$

The analytical solution of the above problem is $y(x) = x(1-x)e^x$. Solution and its derivatives obtained by the proposed method are presented with Figures 1-3. MAE is presented in Table-1.

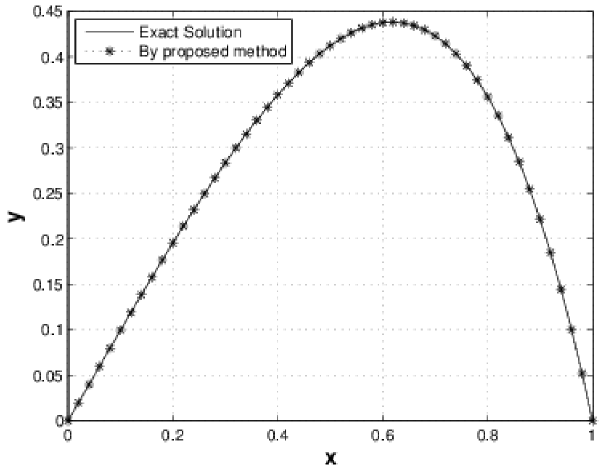

Figure-1. Solution y for example-1.

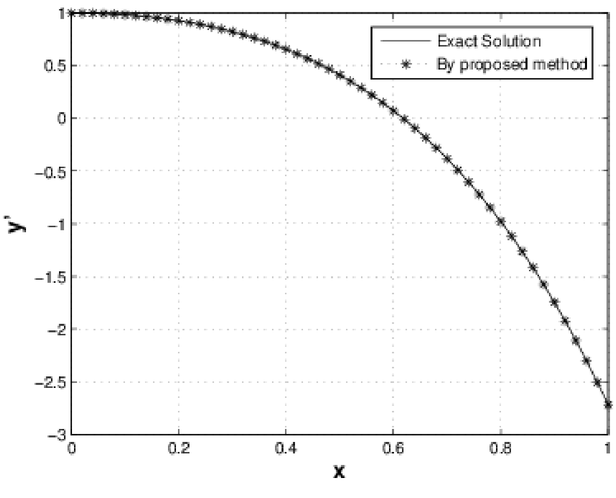

Figure-2. First derivative y' of y for example-1.

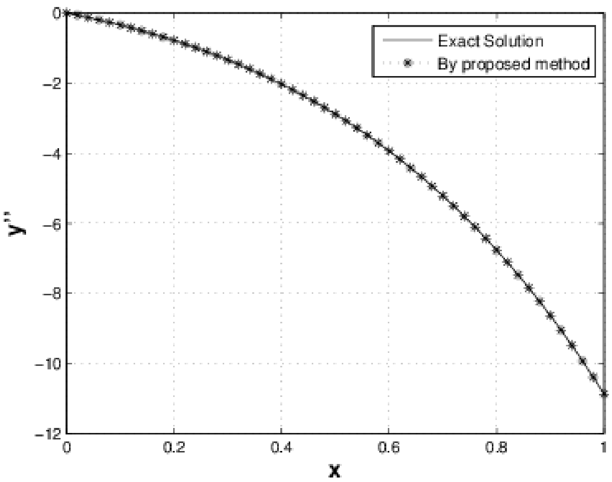

Figure-3. Second derivative y'' of y for example-1.

Table-1. Mean absolute error for example-1.

	$\mathbf{y}$	$\mathbf{y'}$	$\mathbf{y''}$
MAE	4.812×10^{-9}	2.375×10^{-8}	1.813×10^{-7}

Example 2: Consider the following eighth order linear boundary value problem

$$y^{(8)} - y = -8(2x\cos x + 7\sin x) \qquad -1 \le x \le 1$$
$$y(-1) = y(1) = 0 \quad y'(-1) = y'(1) = 2\sin(1)$$
$$-y''(-1) = y''(1) = 4\cos(1) + 2\sin(1)$$
$$y'''(-1) = y'''(1) = 6\cos(1) - 6\sin(1)$$

The analytical solution of the above problem is$y(x) = (x^2 - 1)\sin x$. Solution and its derivatives obtained by the proposed method are presented with Figures 4-6. MAE is presented in Table-2.

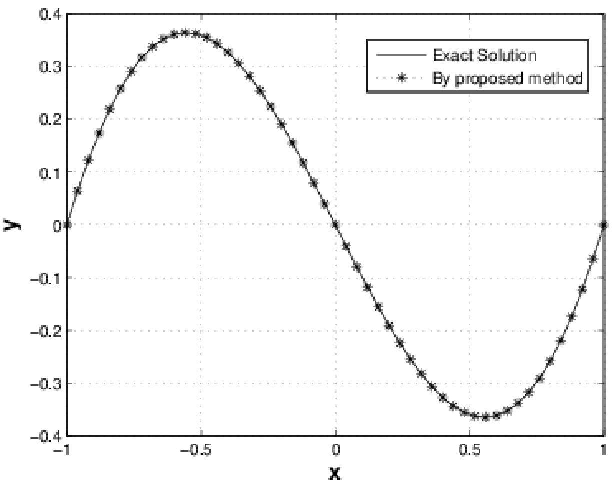

Figure-4. Solution y for example-2.

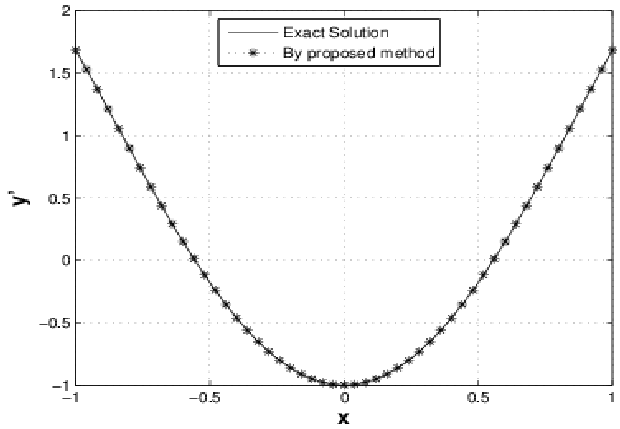

Figure-5. First derivative y' of y for example-2.

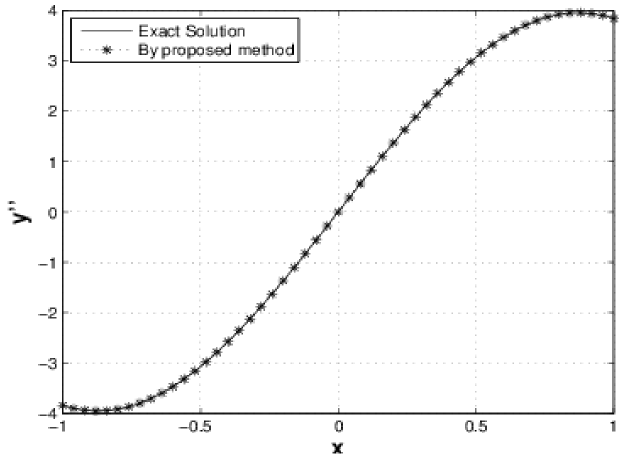

Figure-6. Second derivative y'' of y for example-2.

Table-2. Mean absolute error for example-2.

	y	y'	y''
MAE	9.43×10^{-8}	3.94×10^{-7}	2.068×10^{-6}

7. CONCLUSIONS

In this paper collocation method with nonic B-splines as basis functions is presented to solve eighth order bvp. The proposed method gives a good approximation to the solution of considered bvp and also to its derivatives. Numerical results obtained by the method for the several examples show the efficiency of the proposed method.

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