ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

NEW NUMERICAL STRATEGY TO CALCULATE NORMAL DEPTH IN RECTANGULAR CHANNELS

Myriam Rocío Pallares M.[1] and Wilson Rodríguez C.[2]
[1]Civil Engineering Program, Faculty of Engineering, Surcolombiana University, Colombia
[2] Civil Engineering Program, Faculty of Engineering, Cooperativa University, Colombia
E-Mail: myriam.pallares@usco.edu.co

ABSTRACT

Review, analysis and comparison of explicit and implicit methods of recent development (2009-2015) for the calculation of normal depth in rectangular channels are performed; an adaptation of Newton Raphson method is developed as an alternative calculation of the problem and formulate a new method (called non-orthogonal directions NOD) is presented, this method is based on the modification of the fixed-point iteration method which is recognized for being simple but divergent with application limitations and restrictions. The proposed method determines the start value for the normal depth wisely, its formulation is simple, it has fast convergence and low computational cost that make it competitive with recognized and widely used methods (for example, Newton Raphson) as evidenced by the findings.

Keywords: fixed-point iteration method, newton raphson method, non-orthogonal directions method, normal depth.

INTRODUCTION

When we want to make the hydraulic design of rectangular channels and calculate hydraulic profiles we have to determine the "normal depth". This is done using tables and graphs but they have errors in reading data, interpolation or the accuracy of the scale. We can also work the problem using numerical methods for solving nonlinear equations (closed methods: bisection, false rule or open methods: Newton-Raphson, secant, fixed point iteration); we know too, that these methods share with the problem, the advantages and disadvantages they have in their formulation. Recently, in the literature specialized over hydraulic of channels –these are, high-level books, articles, papers on journals and proceedings– new methods based on explicit calculation of a series of well-defined steps have emerged, if these methods are properly executed they give satisfactory results compared with the order of accuracy required in most engineering calculations. Some proposals of iterative calculation for determining the normal depth in rectangular channels have also appeared, but there are always way to improve them and reformulate them to create new methods to overcome their deficiencies and accelerate convergence. This is a research field where the numerical methods take an important role to advance new proposals. (Nakamura, 1997).

THEORETICAL FRAMEWORK

Equation (1) is called Manning equation and is widely accepted to calculate the flow in a channel when the flow rate is uniform. This equation (in SI units) indicates that,

$$Q=\frac{A_h}{n}R_h^{2/3}S_o^{1/2} \tag{1}$$

where:
Q: is the flow rate, [m^3/ s]
Ah: the hydraulic area, [m^2]
Pm: is the wetted perimeter of the channel, [m]
Rh: is the hydraulic radius defined as Rh=Ah/Pm, [m]
So: is the slope of the channel bottom
n: is the Manning roughness coefficient (dimensionless)

Equation (1) may be ordered to find the expression (2):

$$M_s=\frac{Qn}{S_o^{1/2}}=A_hR_h^{2/3} \tag{2}$$

Equation (2) is a typical expression of numerical methods. In this equation, the term on the right is nonlinear and it is called Section Modulus (Ms). It increases with increasing normal depth, this feature makes the equation has only one solutiong.

SHORT STATE OF THE ART

As we said before, there have emerged explicit and iterative methods for calculating the normal depth in rectangular channels. These have been emerging in the specialized literature. The last appearance of the explicit methods is Vatankhah and Easa (2011), meanwhile, the latest proposal in iterative methods was produced by Knight *et al*. (2009).

a) Vatankhah y Easa (Explicit method)

This method was proposed in 2011. It uses two parameters and a simple equation to determine the normal depth. Their formulation is summarized below (remember, *b* is the width of the rectangular channel and y_n the normal depth, in meters).

1) $\beta_r=\dfrac{Qn}{S_o^{1/2}b^{8/3}}$

2) $\eta_n=\beta_r^{\frac{3}{5}}\left(1+2\beta_r^{\frac{3}{5}}+1.712\beta_r^{\frac{6}{5}}\right)^{\frac{2}{5}}$

3) $y_n=\eta_n b$

b) Knigh *et al*. (Iterative method)

This method was proposed in 2009. It uses an iterative equation for trapezoidal channels. When the case is simplified to a rectangular channel the result is equation (3) (remember, b is the width of the rectangular channel and y_n the normal depth, in meters).

$$y_n^{i+1} = \left(\frac{Qn}{S_o^{1/2}}\right)^{\frac{3}{5}} \frac{\left(b+2y_n^i\right)^{\frac{2}{5}}}{b} \tag{3}$$

The method requires an arbitrary starting value which may even be zero and the stop criterion can be defined in relative or absolute terms; often a stopping criterion in absolute terms is used. In the case of flow in channels we consider enough a tolerance of 1 mm for the normal depth according to the order of magnitude of a typical real channel, such as,

$$\left|y_n^{i+1} - y_n^i\right| \le 0.001 \tag{4}$$

ADAPTATION OF THE NEWTON-RAPHSON METHOD FOR CALCULATING THE NORMAL DEPTH IN A RECTANGULAR CHANNEL

We generate a new iteration formula based on the method of Newton-Raphson (NR), this formula gets all the advantages of its predecessor, mainly the quadratic convergence rate.

If we substitute equations of hydraulic area and hydraulic radius (5) y (6) respectively into equation (2) for a rectangular section we get an equation (7).

$$A_h = (b*y) \tag{5}$$

$$R_h = \frac{b*y}{(b+2y)} \tag{6}$$

Expression (7) is a problem of nonlinear equation roots.

$$f(y) = y\left(\frac{b*y}{b+2*y}\right)^{2/3} - \frac{M_s}{b} = 0 \tag{7}$$

where:
y: is the normal depth in rectangular channels, [m]
b: is the width of the rectangular channel, [m]
y=yn: for the uniform flow.

The Newton-Raphson method requires finding the slope of the function f(y), for this, we use the open source software called MAXIMA to determine the expression (8),

$$f'(y) = \frac{(by)^{2/3}(6y+5b)}{3(b+2*y)^{5/3}} \tag{8}$$

When we simplify the Newton-Raphson equation and arrange it suitably, we can get the equation (9) which is the iterative expression to determine the normal depth in rectangular channels.

$$y_n^{i+1} = y_n^i + \frac{3\left(b+2*y_n^i\right)\left(M_s\left(b+2*y_n^i\right)^{2/3} - \left(b*y_n^i\right)^{5/3}\right)}{b\left(6y_n^i+5b\right)\left(b*y_n^i\right)^{2/3}} \tag{9}$$

In equation (9) the starting value is arbitrary and has to be no-zero as this expression becomes indeterminate when a null value is taken [1, 2, 3, 5, 6].

PROPOSAL FOR A NEW METHOD FOR CALCULATING THE NORMAL DEPHT IN A RECTANGULAR CHANNEL

A new formulation called non-orthogonal direction method "NOD" is proposed. It´s based on a modification of its predecessor called fixed point iteration method.

The distinctive features of the new method are: smart determination of the initial start value, high speed of convergence, low computational cost and simplicity of the formulation. This new method is an original proposal that becomes it in a compete method against the Newton-Raphson method. New method is based on a modification of fixed point iteration method to overcome their problems of divergence –which are typical and that restrict their use–.

Adding the variable "y" on both sides of the equation (7), we can get the typical expression (10) of the iteration fixed point method.

$$g(y_n^i) = y_n^i\left(\frac{b*y_n^i}{b+2*y_n^i}\right)^{2/3} - \frac{M_s}{b} + y_n^i = y_n^{i+1} \tag{10}$$

Equation (10) is divergent; however, it can be sorted and simplified as shown in (11) and (12),

$$y_n^{i+1} = \frac{my_n^i - g\left(y_n^i\right)}{m-1} \tag{11}$$

$$y_n^{i+1} = \left(\frac{1}{m-1}\right)\left[y_n^i\left((m-1) - \left(\frac{b*y_n^i}{b+2*y_n^i}\right)^{2/3}\right) + \frac{M_s}{b}\right] \tag{12}$$

Equation (12) is convergent and it doesn’t represent a greater computational cost compared with equation (10). The search direction factor "m" must be greater than 1. There is a way to calculate a value "m" optimal or near-optimal to accelerate convergence and find the seed value of normal depth near to the solution. This new method has two advantages: a) it suggests a smart way to calculate a seed value obviating the selection of arbitrary values, b) it has a way to calculate the "m" factor with similar or superior to Newton-Raphson convergence, but it doesn´t require the derivative of the function.

The procedure for calculating the normal depth of seed value and the optimal value of "m", is,

1) Calculate the seed value from the equation (7) applying y1* and y2* values. These values must produce change of sign in the function f(y) –this operation is simple because the search space in meters is not big–.
2) Calculate the functions g(y) by adding the value (y) to f(y).
3) Calculate "mr" and "br" values from a straight line drawn between the points (y1*,g(y1*)) and (y2*,g(y2*)). This procedure is numerically feasible and reasonable because the g(y) despite it is a nonlinear function displays a graphical function that looks like a linear function close to the solution –at the intersection of the line y=x and g(y), according to the method of fixed point iteration–, so, using of a line is valid as a first approximation. Then equations (13) and (14) allow calculating the values of "mr" and "br".

$$m_r = \frac{g(y_1^*) - g(y_2^*)}{y_1^* - y_2^*} \tag{13}$$

$$b_r = g(y_1^*) - m_r y_1^* \tag{14}$$

4) Determine the optimum value of factor "m" and seed value of normal depth "yn0" using expressions (15) and (16) respectively.

$$m = m_{opt} = m_r \tag{15}$$

$$y_n^0 = \frac{b_r}{(1 - m_r)} \tag{16}$$

5) Apply equation (12). After one or two iterations we get very satisfactory results for the required accuracy.

RESULTS AND DISCUSSION

To verify the performance of the proposed formulation we use a typical case of calculating the normal depth in a rectangular channel. Channel characteristics are,

Flow rate: Q=4,5 [m^3/s],
Manning roughness coefficient: n=0,013,
Slope: So=0.0018,
Width channel: b=2.0 [m],

The stop criterion is a tolerance of 1 [mm], that is, the absolute error iteration less than 0.001. In addition, we assume that the exact solution is the value of the variable that overrides the function. The value of "yn" verified in both methods (Newton Raphson and NOD) is 1.07030484108076.

The results in Table-1 indicate that the methods are satisfactory; however, the NOD method converges faster to the solution with very small percentage errors. When the method starts with a seed value yo=1.069 obtained from the formulation described in paragraph 5. We can see clear advantages over other methods; however, when the proposed method doesn't use the seed and start at zero also surpasses Newton-Raphson and Knight et. al. because the solution is achieved quickly with a very small relative error (10E-08). Note that Newton-Raphson can't start at zero because the derivative becomes zero and the equation is indeterminate.

Table-1. Benchmarking methods.

Method	Stop iteration	y_n	Relative error vs. Exact value (%)
Vatankhah & Easa	---	1.07105446060914	0.070e+00
Knight et al (y_o=0)	6	1.07019716910401	0.010e+00
Newton-Raphson (y_o=0.01)	5	1.07030484108079	2.803e-12
NOD (y_o=0)	4	1.07030484184407	7.132e-08
NOD (y_o=1.069)	1	1.07030495557215	1.070e-05

From convergence curves in Figure-1, we can deduce that the factor "m" in the NOD method has an optimum value. This value can be obtained from the results of the iterations of the Newton-Raphson method if the derivative value is reviewed in the final solution. That is, if the value of the derivative of Newton is added the unit we can obtain the optimal value "m" which makes the "NOD" method has a quadratic convergence ($m=m_{opt}=1+f'(y)$), where y is evaluated in the final solution of the Newton-Raphson method). That is, NOD method becomes a "degenerate clone" of Newton-Raphson method if different values of "m" are used. This interesting property demonstrates the possibility to get quadratic or quasi-quadratic convergence without having to derive all the time as the NR method. This quality reduces the high costs of operations and complexity in the expressions of iteration. The dotted line in Figure-1 represents the convergence of NOD method based on the value of "m" extracted from the calculations of NR method (m_{opt}= 1.85156559512771). We can see the quadratic convergence and the similarity with the convergence curve of NR method. If we add the search strategy that uses a smart seed, NOD method becomes a highly competitive alternative for solving nonlinear problems.

Figure-1 shows that the convergence of Knight et. al and NOD (y_o=1,069) is linear, however, the slope of NOD method is greater than Knight suggesting better convergence; also it has the advantage of intelligent seed that surpasses all alternatives evaluated (the iterative phase of "NOD" method refines the solution).

When the NOD method starting with yo=0, it outperforms other alternatives, except NR method (at iteration 5 presents a small difference). However, the NOD method reaches tolerance at iteration 4 and maintains advantage over NR.

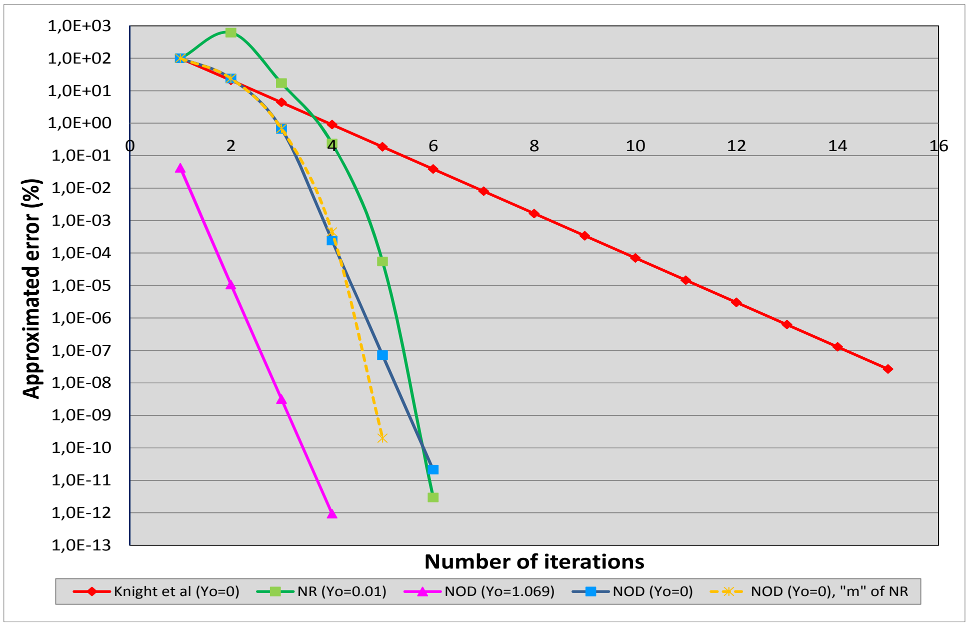

Figure-1. Convergence methods.

CONCLUSIONS

We present a review of recent and interesting methods for calculating the normal depth in rectangular channels and other developed by the authors that are competitive with existing ones.

The proposed NOD method can be called a degenerate clone of Newton-Raphson method (with values of "*m*" different from the optimum value). This new method is very interesting because some values of parameter "m" makes it a perfect copy of NR method, but, the new method can mutate negatively when convergence is lower than the NR method and, it can mutate positively if smart seeds and appropriate values of "*m*".

ACKNOWLEDGEMENTS

Thanks to Surcolombiana University and Cooperativa University for the support given to this research.

REFERENCES

[1] Akai T. J. 2004. Métodos Numéricos Aplicados a la Ingeniería. México: Limusa Willey.

[2] Burden R. L. and Faires D. J. 2011. Analisis Numerico. México: Cengage Learning Editores S.A.

[3] Chapra S. C. and Canale P. R. 2007. Métodos Numéricos para Ingenieros. México: Mc. Graw Hill.

[4] Donald W. K., *et al.*, (2009). Practical Channel hydraulics, London: CRC Press.

[5] Heath M. T. 2002. Scientific Computing: An Introductory Survey. Boston: McGraw-Hill.

[6] Mathews J. H. and Fink K. D. 2011. Metodos Numericos con Matlab. México: Cengage Learning Editores S.A.

[7] Nakamura S. 1997. Análisis Numérico y Visualización Gráfica con Matlab. México: Pearson Education.

[8] Press W. H., Teukolski S. A. and Vetterling W. T. 2012. Numerical Recipes In C: The Art of Scientific Computing. London: Cambridge University Press.

[9] Vatankhah, R.A. and Easa. 2011. Explicit solutions for critical and normal depths in channels with different shapes: Journal of flow measurement and instrumentation. Vol. 22, pp. 43-49.