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PARADIGM FOR NATURAL FREQUENCY OF AN UN-CRACKED CANTILEVER BEAM AND ITS APPLICATION TO CRACKED BEAM

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ABSTRACT

Presence of crack in a beam increases local flexibility; hence dynamics of the structures gets changed to a considerable degree. Crack gets propagated in the material due to fatigue and at the end, it leads to catastrophic failure, hence it needs much attention. Scientific analysis of such phenomena is important because it can be used for crack detection in structures and fault diagnosis. The natural frequency is most important vibration parameter, as it is extensively used as an input for the crack detection by the vibration methods. In the design of the structures or elements, natural frequency plays an important role. In this study, a theoretical method of analysis of the first natural frequency of an un-cracked cantilever beam in a bending mode is presented. The converged natural frequency formula of a paradigm is extended either to a single cracked beam or multiple cracked beam. To get the natural frequency of a cracked beam by a proposed method, vibration parameter such as stiffness is required; therefore in this study; static analysis of a cracked beam is done by using ANSYS software to get the zero frequency deflection. Stiffness of the cracked beam is then calculated by using conventional formula (Load / deflection). This method gives outstanding results for natural frequencies for both single and multiple cracked specimens. Single sided cracks are considered on the beam, as it is very common localized defect and occurred in the beam due to the fatigue load. Modal analysis is done by using ANSYS software to get the natural frequency of intact beam and cracked cantilever beam. The natural frequency obtained by the proposed method for a crack free beam, and beam having either single crack or multiple cracks gives good agreement with the natural frequency obtained by ANSYS. The main attraction of this method is that it gives one more way to the researchers to determine the modal properties of a cracked beam; the only thing is that some additional tools such as simulation software's or experimental methods are required to evaluate cracked beam stiffness.

Keywords: natural frequency, cantilever beam, EN 47, transverse crack, ANSYS, stiffness.

INTRODUCTION

The vibration analysis of a cracked beams and shafts is one of the severe problems in turbo machinery. The investigation of these elements for vibration characteristics is of great interest due to its practical importance. Measurements of natural frequencies, vibration modes are used to predict the location and size of the crack in the beam. Appearance of the cracks on the beam is mainly due to erosion and corrosion phenomena, fatigue strength of the materials. In the past there have been considerable attempts to understand the dynamics of a cracked beam [1-9]. Christides and Barr [1] developed a one-dimensional cracked beam theory at the same level of approximation as the Bernoulli-Euler beam theory. Several assumptions on the displacement, velocity and stress fields are built into this mode. The pair of symmetric cracks is always assumed to remain open as the beam is vibrating, so as to avoid the non-linear characteristics of an opening and closing crack. An approximate Galerkin solution to the one-dimensional cracked beam theory was obtained by Shen and Pierre [2]. The comparison functions used in the Galerkin procedure consisted of mode shapes of an uncracked beam. Shen and Chu [3] extended the cracked beam theory to account for the opening and closing of the crack-the so-called breathing crack model. A Galerkin procedure was used to obtain the bilinear equation for each vibration mode. The non linear dynamic response of the bilinear equation to a forcing excitation was calculated through a numerical analysis. Chu and Shen [4] obtained a closed form solution for a forced single-degree-of-freedom bilinear oscillator under low frequency excitation. They extended the procedure in order to study the dynamics of cracked beams with bilinear forcing functions. Yuen [5] proposed that the change in the stiffness of the cracked beam at the location of the crack can be modelled as a change in the modulus of elasticity of the cracked location. The finite element method was used to carry out the analysis. Shen and Taylor [6] developed an identification procedure for an on-line detection of the size and location of cracks. A mean square difference and a mini-max criterion were used to demonstrate the reliability of the identification procedure. Ostachowicz and Krawczuk [7] replaced the crack section with a spring and then carried out modal analysis for each part of the beam using appropriate matching conditions at the location of the spring. The equivalent stiffness of the spring was calculated using the stress intensity factor at the crack location. Qian *et al.* [8] derived an element stiffness matrix of a beam with a crack, based on the integration of stress intensity factors. The finite element method was used to study the vibration response of the beam. Abraham and Brandon [9] modelled the opening and closing of a crack using a substructuring approach. Lagrange multipliers and time varying connection matrices were used to represent the interaction forces between the two segments of the cantilever beam separated by the crack. The effect of dry friction when the crack is closed has been accounted for in this model. Ostachowich W.M [10] studied the effect of crack locations and sizes on the vibrational behavior of the structure for the forced vibrations. The assumptions of open and closed crack leads to a model with point finite elements. In the case of cracked beams, the crack breathing law is quite simple, since there are only two states for the stiffness matrix: when the crack is open and when it is closed. With this behavior, the stiffness variation is assumed as step function, according to the instantaneous bending moments that is applied to the crack section, as Qian *et al.* [11] and Sundermeyer and Weaver [12] analyzes a simply supported cracked beam, model by two beams segments joined by a spring that represents cracked section. Each segment is treated as a continuous element, which obeys the differential partial equation of Euler-Bernoulli. On the other hand, Tsai and Wang [13] uses the Timoshenko's theory in order to model the beam section. Qian *et al.* [11] formulate a method of crack location in cantilever beams, based on the change that this failure produces in the natural frequencies and mode shapes of the system. Saavedra *et al.* [14] presented a theoretical and experimental dynamic behaviour of different multi-beams systems containing a transverse cracks. The additional flexibility that the crack generates in its vicinity is evaluated using strain energy density function given by the linear fracture mechanics theory. Based on this flexibility, a new cracked finite element stiffness matrix is deduced, which can be used subsequently in the FEM analysis of crack systems. Chaiti *et al.* [15] addresses the problem of vibrations of a cracked beam. In general, the motion of such a beam can be very complex. The focus of this paper is the modal analysis of a cantilever beam with a transverse edge crack. The non-linearity mentioned above has been modelled as a piecewise-linear system. In an attempt to define effective natural frequencies for this piecewise-linear system, the idea of a bilinear frequency is utilized. The bilinear frequency is obtained by computing the associated frequencies of each of the linear pieces of the piecewise-linear system. The finite element method is used to obtain the natural frequencies in each linear region. Chaudhari and Maiti [16] proposed the modeling of transverse vibration of a beam of linearly variable depth and constant thickness in the presence of an open edge crack normal to its axis, using the concept of a rotational spring to represent the crack section and the frobenius method to enable possible detection of location of the crack based on the measurement of natural frequencies. The method can also be used to solve the forward problem. Filiz *et al.* [17] studied the axial vibration of carbon nanotube heterojunctions using nonlocal rod theory. The nonlocal constitutive equations of Eringen are used in the formulations. The carbon nanotubes with different lengths, chirality and diameters are considered in the heterojunctions. Effect of nonlocality, length of the carbon nanotubes and lengths of each segment are investigated in detail for each considered problem. It is obtained that, by joining carbon nanotubes good vibrational properties are obtained by suitable selection of parameters. Mesut smisek [18] studied the forced vibration of a simply supported single-walled carbon nanotube (SWCNT) subjected to a moving harmonic load is investigated by using nonlocal Euler–Bernoulli beam theory. The time-domain responses are obtained by using both the modal analysis method and the direct integration method. The effects of nonlocal parameter, aspect ratio, velocity and the excitation frequency of the moving load on the dynamic responses of SWCNT is discussed. Metin Aydogdu [19] is developed a nonlocal elastic rod model and applied to investigate the small-scale effect on axial vibration of nanorods. Explicit expressions are derived for frequencies for clamped–clamped and clamped–free boundary conditions. It is concluded that the axial vibration frequencies are highly over estimated by the classical (local) rod model, which ignores the effect of small-length scale. Present results can be used for axial vibration of single-walled carbon nanotubes. Sudak L.J [20] is presented the model based on the theory of nonlocal continuum mechanics, on the column buckling of multiwalled carbon nanotubes. The present analysis considers that each of the nested concentric tubes is an individual column and that the deflection of all the columns is coupled together through the van der Waals interactions between adjacent tubes. Based on this description, a condition is derived in terms of the parameters that describe the van der Waals forces and the small internal length scale effects. In particular, an explicit expression is derived for the critical axial strain of a double walled carbon nanotube which clearly demonstrates that small scale effects contribute significantly to the mechanical behavior of multiwalled carbon nanotubes.

Rizos and Aspragathos studied [21] the cracked cantilever bam with rectangular cross-section for harmonic excitation. The beam is forced by a harmonic vibration exciter to vibrate at one of the natural frequency. From the measured amplitudes at the two points of the structure vibrating at one of its natural modes, the respective vibration frequency and an analytical solution of the dynamic response, the crack location can be found and depth can be estimated with reasonable accuracy. Zheng and Fan [22] presents simple tools for the vibration and stability analysis of cracked hollow-sectional beams. It consists of two parts. In the first, the influences of sectional cracks are expressed in terms of flexibility induced. Each crack is assigned with a local flexibility coefficient, which is derived by virtue of theories of fracture mechanics. The flexibility coefficient is a function of the crack depth. The second is for deeper penetration, in which the crack goes into the center hollow-sectional region. Least-squares methods are used to generate the explicit formulae and are best-fitted equations. The best-fitted curves are presented. From the curves, the flexibility coefficients can be read out easily, while the explicit expressions facilitate easy implementation in computer analysis. From the curves, the flexibility coefficients can be read out easily, while the explicit expressions leads to easy implementation in computer analysis. Pandey *et al.* [23] investigated a new parameter called curvature mode shape as a possible contender for identifying and locating the damage in a structure. In this study cantilever and simply supported beam are considered to predict the location and size of the crack. The change in the curvature mode shapes is found more as size of the damage

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increases. This information is needed to obtain the amount of damage in the beam. Finite element analysis was used to get the displacement mode shapes of the two models. Curvature mode shapes were then calculated from displacement mode shapes by using a central difference approximation. Turgut and Mesut [24] analyzed free vibration of Timoshenko beams having different boundary conditions. For examining the free vibration characteristics of Timoshenko beams, Lagrange equations are used. The first eight natural frequencies of Timoshenko beam are calculated and tabulated for different thickness-to-length ratios. From this study, it is concluded that the tabulated results will show useful to designers and provide a reference against which other researchers can evaluate their results. Kocatürk and Simsek [25] is investigated the free vibration of elastically supported beams based on Timoshenko beam theory. The free vibration characteristics of Timoshenko beams are examined by Lagrange equations. The first three natural frequencies of the Timoshenko beams are calculated for various rigidity values of translational and rotational springs, and obtained results are not only put into a table, but also presented in three-dimensional plots. It is considered that the tabulated results will show useful to designers and give a reference against which other researchers can compare their results. Lee and Schultz [26] presented a study of the free vibration of Timoshenko beams and axisymmetric Mindlin plates. The analysis is based on the Chebyshev pseudospectral method. This method is widely used in the solution of fluid mechanics problems. Simply supported, clamped, free and sliding boundary conditions of Timoshenko beams are treated, and numerical results are shown for different thickness-to-length ratios. Lee *et al.* [27] investigated the free vibration problem of Timoshenko beams with an internal hinge. Accurate vibration frequencies for axially loaded, clamped-simply supported beams and clamped-clamped beams are determined. The effects of transverse shear deformation, axial force, rotary inertia, and the location of the internal hinge on the fundamental frequency of vibration are investigated. A necessary situation for the best possible location of internal hinge that maximizes the fundamental frequency is also presented. Nallim and Grossi [28] present a simple, accurate and flexible general algorithm for the study of a large number of beams vibration problems. The approach is developed based on the Rayleigh–Ritz method with characteristic orthogonal polynomial shape functions. It allows the inclusion of a number of complicating effects such as varying cross-sections, ends elastically restrained against rotation and translation and presence of an axial, tensile force. A number of cases are treated to show the simplicity and great flexibility of this approach, in the determination of frequencies. Simsek [29] analyzed the free vibration of beams subjected to axial loads and having different boundary conditions by Bernoulli-Euler beam theory. The free vibration characteristic of the beams is examined by Lagrange equations. By using very stiff linear spring constants the constraint conditions of the supports are taken into account. Trial functions denoting the deflection of the beam is expressed in the form of power series to apply the Lagrange equations. It is measured that the tabulated results will prove practical to designers and provides a reference against which other researchers can evaluate their results. Virgin and Plaut [30] considered the steady state linear response of beams subjected to a distributed, harmonically varying, transverse force. A static axial load, either compressive or tensile, is applied to the beam, and damping is included in the analysis. The response of the beam for the axial load is investigated, with attentiveness focused on the maximum amplitude of the central deflection over all attainable forcing frequencies. Detection of the crack presence on the surface of beam-type structural element using natural frequency is presented by Barad *et al.* [31]. Experimentally first two natural frequencies of the cracked beam have been obtained and used for detection of crack location and size. Predicted crack locations and size are compared with the actual results and found to be in good agreement. Collins *et al.* [32] investigated the longitudinal vibrations of a cantilever bar with a transverse crack. The frequency spectra are computed and the effects of crack location and compliance on the fundamental natural frequency are determined. For vibration caused by harmonic force, the steady state amplitude of motion of the free end is plotted as a function of forcing frequency. Crack compliance and crack location. Breathing crack results are compared to that crack which remains open. Aydogdu [33] is developed a nonlocal elastic rod model and applied to investigate the small-scale effect on axial vibration of nanorods. For different boundary conditions like clamped–clamped and clamped–free boundary conditions, explicit expressions are derived for frequencies. It is observed that the axial vibration frequencies are highly over estimated by the classical (local) rod model, which ignores the effect of small-length scale. For axial vibration of single-walled carbon nanotubes, the present results can be used. Simsek [34] investigated the forced vibration of a simply supported single-walled carbon nanotube subjected to a moving harmonic load by using nonlocal Euler–Bernoulli beam theory. By using both the modal analysis method and the direct integration method, the time-domain responses are obtained. The effects of nonlocal parameter, velocity, aspect ratio and the excitation frequency of the moving load on the dynamic responses of SWCNT are discussed. The results show that load velocity and the excitation frequency play an important role on the dynamic behaviour of the SWCNT. Mungla *et al.* [35] measured the natural frequencies of cracked and uncracked clamped-clamped beam by using single-input multi-output (SIMO) based experimental modal analysis (EMA). The cracks are artificially produced at various locations on the beam using wire cut electro discharge machining process. The measured frequencies of the beam with crack are used to identify crack location and its severity by the frequency-based approach as well as genetic algorithm (GA) based intelligent search for the same test points. Babu and Sekhar [36] have presented the dynamics and diagnostics of cracked rotors have been gaining importance in present years. In many cases, shaft carries multiple cracks. In this study, a new technique amplitude deviation curve (ADC) or slope deflection curve (SDC) has been developed, which is a modification of the operational deflection shape (ODS). The effectiveness of the SDC over ODS for small crack detection has been shown. Neilson and MacConnel [37] have conducted experimental tests to study the vibration characteristics in two different cracks in a long rotor shaft, a notch cut to varying depths and actual crack growth from a pre-crack. The approach was to set up experimental apparatus, develop vibration detection system, and maximize the dynamic range. Fatigue crack initiation and propagation in a pre-cracked high carbon steel shaft was experimentally evaluated and monitored using a vibration based condition monitoring method. The results of the test and analysis clearly demonstrate the feasibility of using vibration to detect the change in frequency of a shaft due to change in stiffness such as those associated with a shaft crack. Singh and Tiwari [38] have presented a two stage identification methodology, which identify the number of cracks and crack parameters. The methodology uses transverse forced response of a shaft system at different frequencies of a harmonic excitation. In the first stage, a multi-crack detection and its localization algorithm are developed. In the second stage of the algorithm, the size and the accurate location of cracks are obtained by using multi-objective genetic algorithms. Responses of the shaft at several frequencies are used to define objective functions in genetic algorithms. Pennacchi [39] have proposed a model -based transverse crack identification method suitable for industrial machines. The method is validated by experimental results obtained on the large test rig, which was expressly designed for investigating the dynamical behaviour of cracked horizontal rotors. The excellent accuracy obtained in identifying crack parameters. Sekhar [40] have presented a method for online identification of cracks in a rotor. The fault-induced change of the rotor system is taken into account by equivalent loads in the mathematical model. The equivalent loads are virtual forces and moments acting on the linear undamaged system to generate a dynamic behaviour identical to the measured one of the damaged system. The rotor has been modelled using finite element method, while the crack is considered through local flexibility change. Lee [41] presents a simple method to identify the multiple cracks in a beam by vibration amplitudes. Crack is taken into account by considering the rotational springs and finite element method is used to solve the forward problem. Inverse problem is solved iteratively by Newton-Raphson method and singular wave decomposition method to get the crack parameters. An improved torsional stiffness model leads to give accurate estimation in crack size. Agarwalla and Parhi [42] analysed the effect of an open crack on the modal parameters of the cantilever beam subjected to free vibration and the results obtained from the numerical method and the experimental method are compared. It is concluded that the structure vibrates with more frequency in the presence of a crack away from the fixed end. Liang *et al.* [43] used the concept of receptance for a system linked with two coordinates (axial and rotational coordinates) for nonuniform beams. Numerical experiments involving the use of a FEA program SAP, to determine the natural frequencies of both uniform and nonuniform beams with a variety of damage cases are used to validate the derived theoretical relationships. Two significant assumptions are concerned in the derived relationship, one is that the structure is considered to behave linearly, and the other one is that the elastic properties of the structure member are time-invariant. Sasmal and Ramanjaneyulu [44] present a methodology for detection and quantification of structural damage using modal information obtained from transfer matrix technique. Damage is assumed at a particular segment of the beam-like structure, an iterative procedure has been formulated to converge the calculated and measured frequencies by tuning the flexural rigidity of elements. It is found that even though the developed methodology is iterative, computational effort is reduced significantly because of use of transfer matrix technique.

From the detailed literature survey, it has been observed that there is no alternative theoretical method to estimate natural frequency of a cracked cantilever beam. Up to now none of the author presented any theoretical method which can be used to determine the bending natural frequency of a cracked cantilever beam. Most of the researchers have investigated the effect of crack locations and crack depths on different vibrating properties like natural frequency, damping factor and resonance amplitude. Investigation of the effect of crack parameters on the stiffness of a cracked cantilever beam of square cross section is also missing in the literature. Also the vibration studies on most practical EN 47 structural material beam is not found in the literature. In the present study, theoretical method is proposed for analysis of the natural frequency of an uncracked cantilever beam and it is applied to the cracked beam. Natural frequency has most significance in the inverse problem and hence there should be some alternative to evaluate the modal properties of the cracked beam. Effect of transverse cracks on vibration parameters i.e. stiffness, natural frequency is investigated by finite element method.

2. MATERIALS AND METHODS

2.1 Material and geometric properties

In this study, EN 47 material is considered. Mass of intact beam and cracked beam are assumed to be same because mass of intact beam is 1123.2 gm and mass of a cantilever beam with two largest depth cracks is 1120.704 gm, hence for crack free and cracked cases of beam mass of beam is taken as 1.1232 kg. The material and geometric properties are given in Table-1. The following material properties are tested in ELCA lab, Pune, India.

Table-1. Material properties and geometric properties of specimen.

Property	EN 47
Density (kg/m^3)	7800

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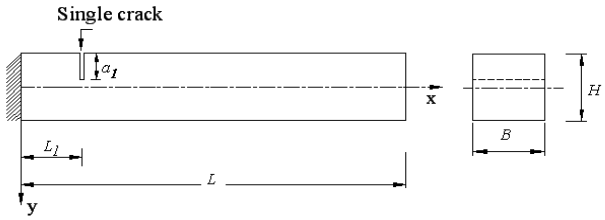
Modulus of Elasticity (N/m^2)	1.95*10^{11}
Mass of specimen (kg)	1.1232
Length of specimen (m)	0.360
Cross Section of specimen (m^2)	0.02*0.02

2.2. Crack configurations

Total 33 specimens are used in this study, out of 33, 1 specimen is crack free and 32 specimens have cracks, to find out how the cracks affect the dynamic behaviour of a cantilever beam. The main case is divided into two cases, case 1 and case2.

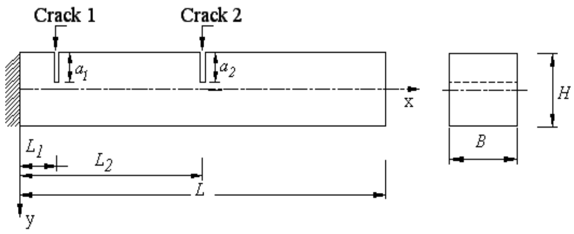
Case 1 details: In this case, 20 specimens of EN 47 material are considered and crack is taken on the specimen from top side as shown in Figure-1. Single transverse crack is taken on each specimen. This case is divided into 5 sub cases. In the first sub case, 60 mm crack location is selected from the cantilever end and at this location crack depth is varied by an interval of 4 mm from 4 mm to 16 mm. The second, third, fourth and fifth sub cases are similar to first sub case, the only difference is that instead of 60 mm crack location, 120 mm, 180mm, 240 mm, and 300 mm crack location is chosen for the second, third, fourth and fifth sub case respectively.

Case 2 details: In this case, 12 specimens of EN 47 material are considered and cracks are taken on the specimen from top side. Two transverse cracks are taken on each specimen as shown in Figure-2. This case is divided into 3 sub cases. In the first sub case, for the first crack 60 mm and for the second crack 120 mm crack location is selected from the cantilever end and at this location crack depth is varied by an interval of 4 mm from 4 mm to 16 mm. Second and third sub case is similar to first sub case the only difference is that for the second crack instead of 120 mm crack location, 180 mm and 240 mm crack locations are taken respectively.



Cantilever beam with a single transverse crack.

Figure-1. Schematic diagram of a cracked cantilever beam.



Cantilever beam with two transverse cracks.

Figure-2. Schematic diagram of a cracked cantilever beam.

2.3 Finite element modelling and analysis

ANSYS [45] finite element program is used to determine natural frequencies of the undamaged as well as cracked beams. For this purpose, rectangle area is created. This area is extruded in the third direction to get the 3 D model. Then at the required location, small rectangular area of crack of 0.5 mm width and required depth is created and extruded. Then small volume of crack is subtracted from large volume of cantilever beam to obtain cracked three dimensional models. The width of crack is kept constant throughout its depth in this study. A 20 node structural solid element (solid 186) is selected for modelling the beam. Finite element boundary conditions are applied on the beam to constrain all degrees of freedom of the extreme left hand end of the beam. Static and modal analyses are carried out on each specimen to get zero frequency deflection and natural frequency. In static analysis 100 N loads is applied at the end of a cantilever beam.

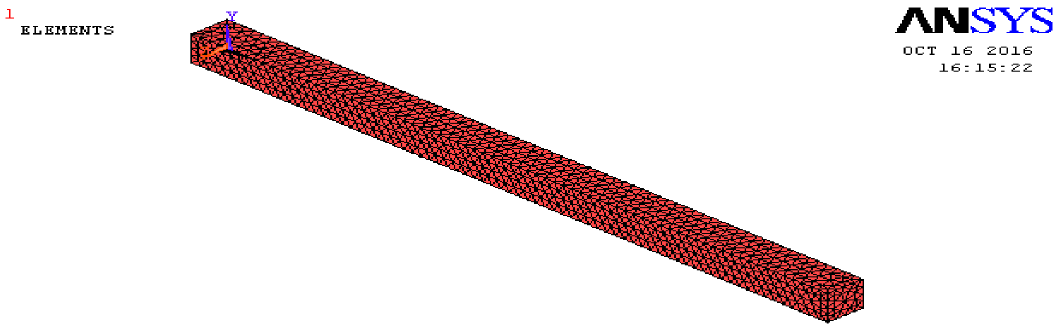


Figure-3. Intact beam finite element modelling.

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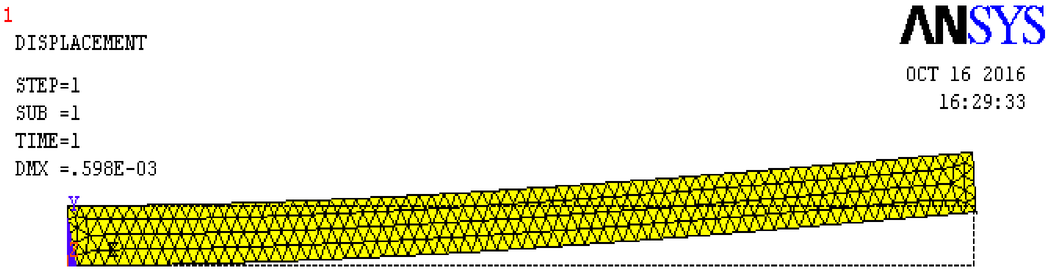

Figure-4. Zero frequency deflection plot of an intact cantilever beam.

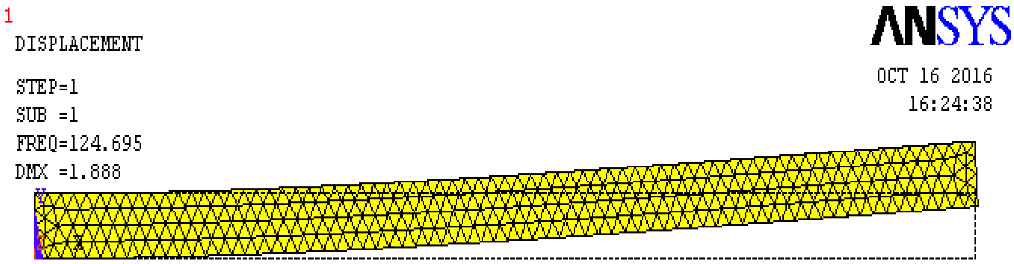

Figure-5. Natural frequency plot of an intact cantilever beam.

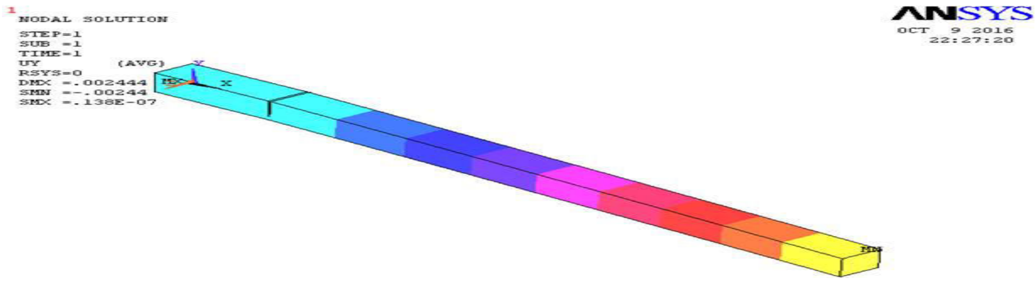

Figure-6. Zero frequency deflection plot, EN 47 TS crack, crack details: L_1/L= 0.166; a/H= 0.8.

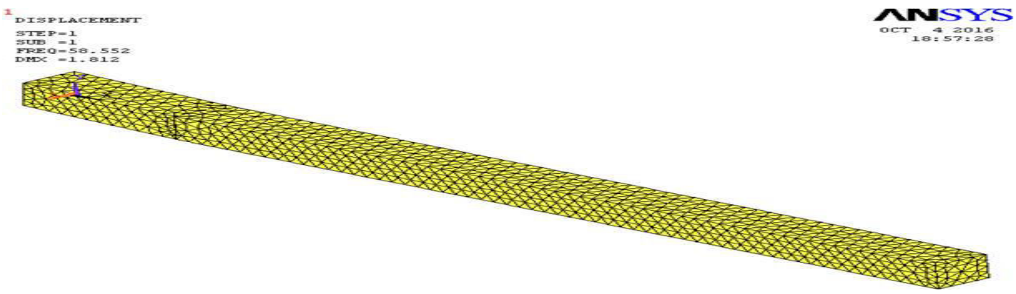

Figure-7. Natural frequency plot, crack details: L_1/L= 0.166; a/H= 0.8.

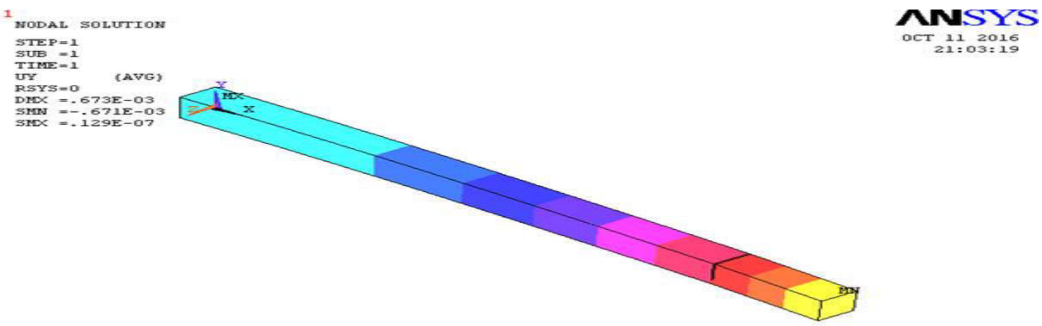

Figure-8. Zero frequency deflection plot, crack details: L_1/L= 0.833; a/H= 0.8.

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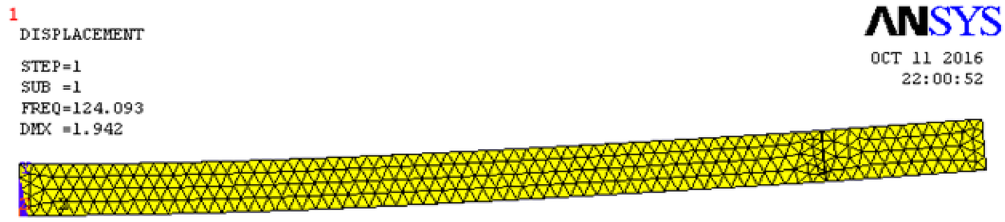

Figure-9. Natural frequency plot, crack details: L_1/L= 0.833; a/H= 0.8.

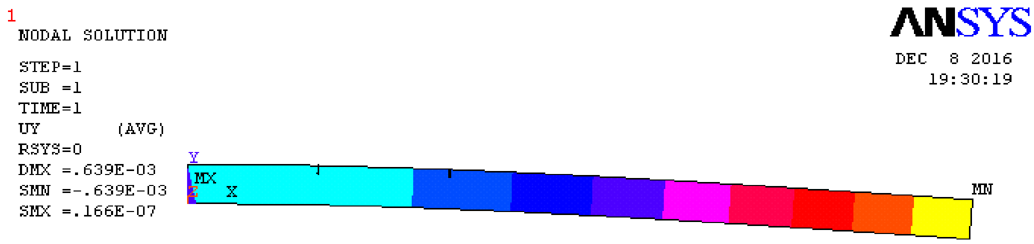

Figure-10. Zero frequency deflection plot, EN 47 TS crack, crack details: L_1/L= 0.166; L_2/L= 0.333; a_1/H= 0.2; a_2/H= 0.2.

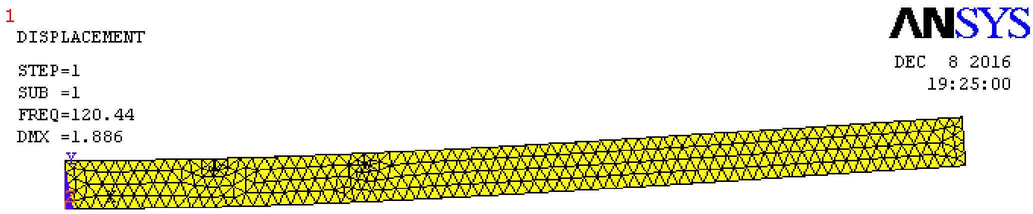

Figure-11. Natural frequency plot, EN 47 TS crack, crack details: L_1/L= 0.166; L_2/L= 0.333; a_1/H= 0.2; a_2/H= 0.2.

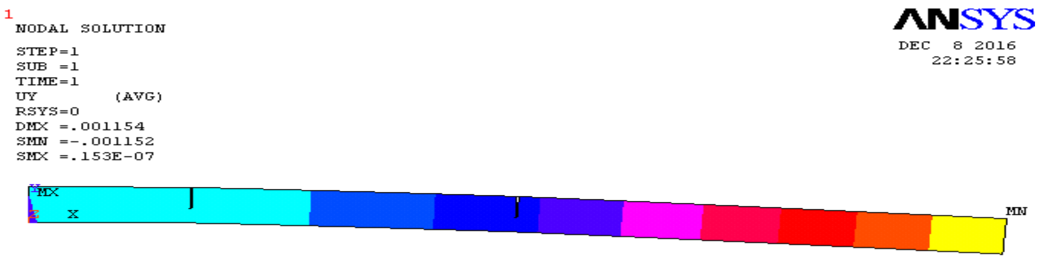

Figure-12. Zero frequency deflection plot, EN 47 TS crack, crack details: L_1/L= 0.166; L_2/L= 0.5; a_1/H= 0.6; a_2/H= 0.6.

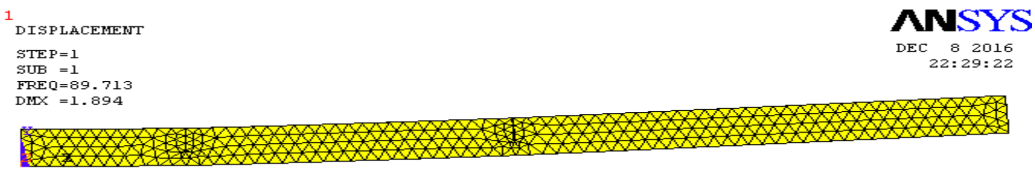

Figure-13. Natural frequency plot, EN 47 TS crack, crack details: L_1/L= 0.166; L_2/L= 0.5; a_1/H= 0.6; a_2/H= 0.6.

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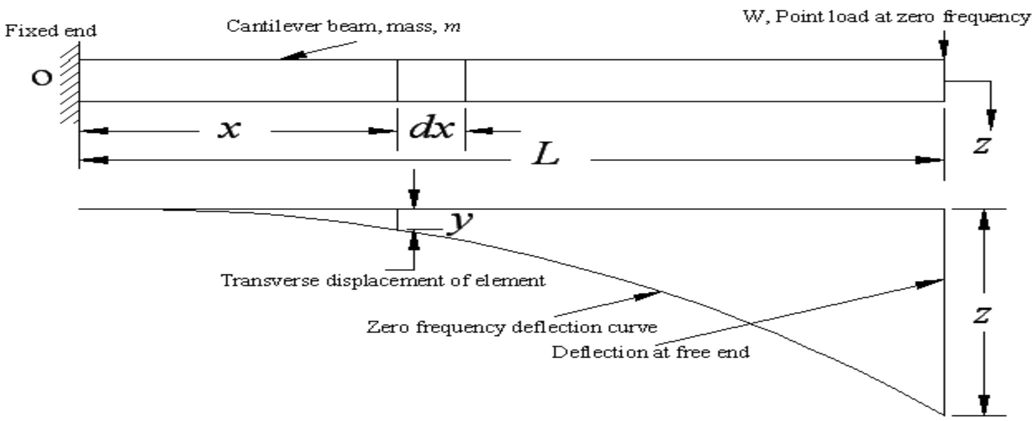

Figure-14. Schematic diagram of a cantilever beam subjected to zero frequency point load.

2.4 Theory

The schematic diagram of a cantilever beam subjected to zero frequency point load is as shown in Figure. 14.

By comparing deflection curve of Figure-4 and Figure-5, it is observed that the deflection curve of a cantilever beam at zero frequency is approximately similar with the curve obtained during vibration with first natural frequency.

Let $\rho'$and L be the mass of the beam per unit length and length of the beam respectively,

Mass of cantilever beam, $m = \rho' * L$

Consider a small element dx at a distance x from the fixed point of a cantilever beam. The mass of the element is dm.

$dm = \rho' dy$

Consider a small element dx at a distance x from the fixed end.

$M_x = -W(L-x)$

$EI\dfrac{d^2y}{dx^2} = -M_x = W(L-x)$

Integrating above equation,

$$EI\frac{dy}{dx} = \int W(L-x)dx = W\left(Lx - \frac{x^2}{2}\right) + C_1 \quad (1)$$

where C_1 is integration constant.

At $x = 0$, $\frac{dy}{dx} = 0$

Substituting above boundary conditions in Equation. (1)

$C_1 = 0$

Substituting value of C_1 in Equation. (1)

$$EI\frac{dy}{dx} = W\left(Lx - \frac{x^2}{2}\right) \quad (2)$$

The Equation. (2) gives the slope value at any section X

Integrating Equation. (2), for a second time

$$EI\, y = W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2 \quad (3)$$

Where C_2 is integration constant?

At $x = 0$, $y = 0$

Substituting above boundary conditions in Equation. (3)

$C_2 = 0$

Substituting value of C_2 in Equation. (3)

$$EI\, y = W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

$$EI\, y = \frac{Wx^2}{6}(3L - x) \quad (4)$$

We have, Deflection at free end, $z = \frac{WL^3}{3EI}$

$W = \dfrac{3EIz}{L^3}$

Substituting value of W in Equation. (4)

$$EI\, y = \frac{3EIz}{L^3}\frac{x^2}{6}(3L - x)$$

$$y = x^2\frac{(3L-x)\,z}{2L^3} \quad (5)$$

Equation. (5) gives the displacement of the element in transverse direction at a distance of x from the fixed end.

Therefore, velocity of the element,

$$y' = x^2\frac{(3L-x)\,z'}{2L^3} \quad (6)$$

The system has kinetic energy due to mass of the cantilever beam

$$\text{T, Kinetic energy} = \frac{1}{2}\rho' dx \int_0^L \left[\frac{x^2(3L-x)z'}{2L^3}\right]^2$$

$$\mathrm{T} = \frac{1}{2}\left(\frac{\rho'}{4\mathrm{L}^6}\right)\mathrm{z}'^2\int_0^L x^4(9L^2 - 6Lx + x^2)\,dx$$

$$\mathrm{T} = \frac{1}{2}\left(\frac{\rho'}{4\mathrm{L}^6}\right)\mathrm{z}'^2\int_0^L (9L^2x^4 - 6Lx^5 + x^6)\,dx$$

$$\mathrm{T} = \frac{1}{2}(0.2357m)\mathrm{z}'^2$$

The potential energy of the cantilever beam

$$V, Potential\ energy = \frac{1}{2}K\,z^2$$

U, Total energy = Kinetic energy + Potential energy

$$U = \frac{1}{2}(0.2357m)\mathrm{z}'^2 + \frac{1}{2}K\,z^2$$

Differentiate above equation with respect to t

$$z'' + \frac{Kz}{(0.2357m)} = 0 \tag{7}$$

$$z'' + \omega_n^2 z = 0 \tag{8}$$

Compare Equation. (7) with standard Equation. (8)

$$\omega_n^2 = \frac{K}{0.2357m}$$

We have, $2\pi f_n = \omega_n$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{0.2357m}}$$

Equation (10) is used to find the natural frequency of intact or a cracked cantilever beam.

3. RESULTS AND DISCUSSIONS

Theoretical values of natural frequencies are estimated by a proposed method. Some ANSYS zero frequency deflection plots are shown in Figure-4, Figure-6, Figure-8, Figure-10, and Figure-12. These plots are required to determine the stiffness of intact and various cracked cases of a cantilever beam.

Table-2. Zero frequency deflection of a cantilever beam with single crack, m.

Location, mm / **Depth, mm**	**L_1 = 60**	**L_1 =120**	**L_1 =180**	**L_1 =240**	**L_1 =300**
$a_1 = 4$	$0.623 * 10^{-3}$	$0.614 * 10^{-3}$	$0.607 * 10^{-3}$	$0.602 * 10^{-3}$	$0.599 * 10^{-3}$
$a_1 = 8$	$0.719 * 10^{-3}$	$0.675 * 10^{-3}$	$0.641 * 10^{-3}$	$0.603 * 10^{-3}$	$0.603 * 10^{-3}$
$a_1 = 12$	$0.998 * 10^{-3}$	$0.852 * 10^{-3}$	$0.746 * 10^{-3}$	$0.663 * 10^{-3}$	$0.614 * 10^{-3}$
$a_1 = 16$	0.00244	0.001778	0.001262	$0.889 * 10^{-3}$	$0.671 * 10^{-3}$

Table-3. Stiffness of a cantilever beam with single crack, N/m.

Location, mm / **Depth, mm**	**L_1 = 60**	**L_1 = 120**	**L_1 = 180**	**L_1 = 240**	**L_1 = 300**
$a_1 = 4$	160513.64	162866.44	164744.64	166112.95	166944.90
$a_1 = 8$	139082.05	148148.14	156006.24	165837.47	165837.48
$a_1 = 12$	100200.4	117370.89	134048.25	150829.56	162866.45
$a_1 = 16$	40983.6	56242.96	79239.30	112485.93	149031.29

Table-4. The first natural frequency of an intact beam.

S. No.	**Methods**	**Natural frequency, Hz**	**% Deviation**
01	Proposed theoretical method	126.49	1.42
	ANSYS	124.69	

Table-5. The first natural frequency of a cantilever beam with single crack, location, L_1 = 60 mm.

L_1 = 60 mm	**Methods**	**Natural frequency, Hz**	**% Deviation**
a_1 = 4 mm	Proposed theoretical method	123.94	1.71
	ANSYS	121.81	
a_1 = 8 mm	Proposed theoretical method	115.37	2.6
	ANSYS	112.36	
a_1 = 12 mm	Proposed theoretical method	97.92	4.21
	ANSYS	93.79	
a_1 = 16 mm	Proposed theoretical method	62.63	6.51
	ANSYS	58.55	

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Table-6. The first natural frequency of a cantilever beam with single crack, location, L_1 = 120 mm.

L_1 = 120 mm	Methods	Natural frequency, Hz	% Deviation
a_1 = 4 mm	Proposed theoretical method	124.84	1.32
	ANSYS	123.19	
a_1 = 8 mm	Proposed theoretical method	119.07	0.85
	ANSYS	118.05	
a_1 = 12 mm	Proposed theoretical method	105.98	0.075
	ANSYS	105.90	
a_1 = 16 mm	Proposed theoretical method	73.37	-2.09
	ANSYS	74.91	

Table-7. The first natural frequency of a cantilever beam with single crack, location L_1 = 180 mm.

L_1 = 180 mm	Methods	Natural frequency, Hz	% Deviation
a_1 = 4 mm	Proposed theoretical method	125.56	1.13
	ANSYS	124.13	
a_1 = 8 mm	Proposed theoretical method	122.19	0.15
	ANSYS	122	
a_1 = 12 mm	Proposed theoretical method	113.26	-2.63
	ANSYS	116.24	
a_1 = 16 mm	Proposed theoretical method	87.08	-9.5
	ANSYS	95.36	

Table-8. The first natural frequency of a cantilever beam with single crack, location L_1 = 240 mm.

L_1 = 240 mm	Methods	Natural frequency, Hz	% Deviation
a_1 = 4 mm	Proposed theoretical method	126.08	1.18
	ANSYS	124.58	
a_1 = 8 mm	Proposed theoretical method	125.98	1.5
	ANSYS	124.09	
a_1 = 12 mm	Proposed theoretical method	120.14	-2.02
	ANSYS	122.57	
a_1 = 16 mm	Proposed theoretical method	103.75	-10.99
	ANSYS	115.16	

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Table-9. The first natural frequency of a cantilever beam with single crack, location L_1 = 300 mm.

L_1 = 300 mm	Methods	Natural frequency, Hz	% Deviation
a_1 = 4 mm	Proposed theoretical method	126.40	1.32
	ANSYS	124.73	
a_1 = 8 mm	Proposed theoretical method	125.98	0.99
	ANSYS	124.73	
a_1 = 12 mm	Proposed theoretical method	124.85	0.16
	ANSYS	124.65	
a_1 = 16 mm	Proposed theoretical method	119.42	-3.91
	ANSYS	124.09	

For the intact beam error between the proposed theoretical method and numerical method for the natural frequency is only 1.42%, it means that proposed theoretical method for natural frequency gives good agreement with the numerical results as shown in Table-4. From Table 4-6, it is found that when crack location ratio remains below 0.5 (below 180 mm); then found error between the theoretical method and numerical method; for the natural frequency is either equal to 6.52% or less than 6.52 %. It means that proposed method gives excellent results for the natural frequency. From Tables 6-8, it is found that when crack location ratio either equal to 0.5 or more than 0.5, then found error for the natural frequency remains very less. 2 cases gives more error, the cracked case which has 180 mm crack location and 16 mm crack depth gives -9.5% errors and the cracked case which has 240 mm crack location and 16 mm crack depth gives -10.99% error.

Table-10. Zero frequency deflection of a cantilever beam with two cracks, m.

Location, mm / Depth, mm	L_1= 60, L_2= 120	L_1= 60, L_2= 120	L_1= 60, L_2= 120
$a_1 = a_2$= 4	$0.639 * 10^{-3}$	$0.632 * 10^{-3}$	$0.627 * 10^{-3}$
$a_1 = a_2$= 8	$0.794 * 10^{-3}$	$0.763 * 10^{-3}$	$0.737 * 10^{-3}$
$a_1 = a_2$= 12	0.001268	0.001152	0.001067
$a_1 = a_2$= 16	0.003604	0.00308	0.002726

Table-11. Stiffness of a cantilever beam with two cracks, N/m.

Location, mm / Depth, mm	L_1= 60, L_2= 120	L_1= 60, L_2= 120	L_1= 60, L_2= 120
$a_1 = a_2$= 4	156494.53	158227.84	159489.63
$a_1 = a_2$= 8	125944.58	131061.59	135685.21
$a_1 = a_2$= 12	78864.35	86805.55	93720.71
$a_1 = a_2$= 16	27746.94	32467.53	36683.79

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Table-12. The first natural frequency of a cantilever beam with two cracks, location, L_1 = 60 mm, L_2= 120 mm.

L_1 = 60 mm, L_2 = 120 mm	Methods	Natural frequency, Hz	% Deviation
$a_1 = a_2 =$ 4 mm	Proposed theoretical method	122.36	1.57
	ANSYS	120.44	
$a_1 = a_2 =$ 8 mm	Proposed theoretical method	109.77	2.04
	ANSYS	107.53	
$a_1 = a_2 =$ 12 mm	Proposed theoretical method	86.86	2.73
	ANSYS	84.49	
$a_1 = a_2 =$ 16 mm	Proposed theoretical method	51.48	3.38
	ANSYS	49.74	

Table-13. The first natural frequency of a cantilever beam with two cracks, location, L_1 = 60 mm, L_2= 180 mm.

L_1 = 60 mm, L_2 = 180 mm	Methods	Natural frequency, Hz	% Deviation
$a_1 = a_2 =$ 4 mm	Proposed theoretical method	123.04	1.43
	ANSYS	121.27	
$a_1 = a_2 =$ 8 mm	Proposed theoretical method	111.98	1.43
	ANSYS	110.37	
$a_1 = a_2 =$ 12 mm	Proposed theoretical method	91.14	1.57
	ANSYS	89.71	
$a_1 = a_2 =$ 16 mm	Proposed theoretical method	55.74	1.65
	ANSYS	54.82	

Table-14. The first natural frequency of a cantilever beam with two cracks, location, L_1 = 60 mm, L_2= 240 mm.

L_1 = 60 mm, L_2 = 240 mm	Methods	Natural frequency, Hz	% Deviation
$a_1 = a_2 =$ 4 mm	Proposed theoretical method	123.53	1.48
	ANSYS	121.7	
$a_1 = a_2 =$ 8 mm	Proposed theoretical method	113.94	1.71
	ANSYS	111.99	
$a_1 = a_2 =$ 12 mm	Proposed theoretical method	94.7	2.09
	ANSYS	92.72	
$a_1 = a_2 =$ 16 mm	Proposed theoretical method	59.24	2.62
	ANSYS	57.69	

Tables 12- 14 shows the natural frequency of a cracked cantilever beam for 12 cracked cases. All the 12 cracked specimens carry 2 transverse cracks from the top side of the beam. Out of 12 cracked cases, one case (a_1= 8 mm, a_2= 8mm, L_1= 60 mm, L_2= 180 mm) is as shown in Figure-2. Found error for the natural frequency between proposed theoretical method and ANSYS is very less (either 3.38% or less than that). This method gives outstanding results for the natural frequency; when beam carries two cracks on the top side as compared to the cantilever beam which carries single edge crack; on the other hand results obtained by the proposed method is valid for single edge and double edge cracked beam.

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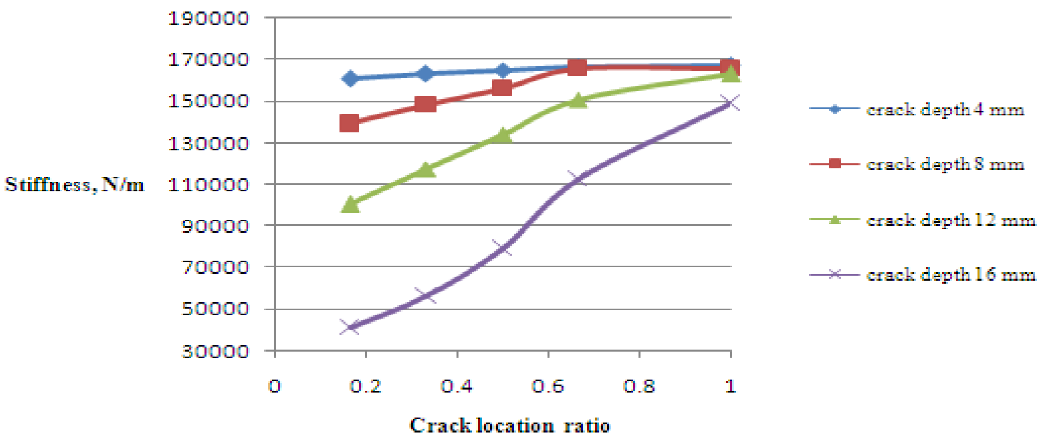

Figure-15. Variation of stiffness versus crack location ratio for single cracked specimen.

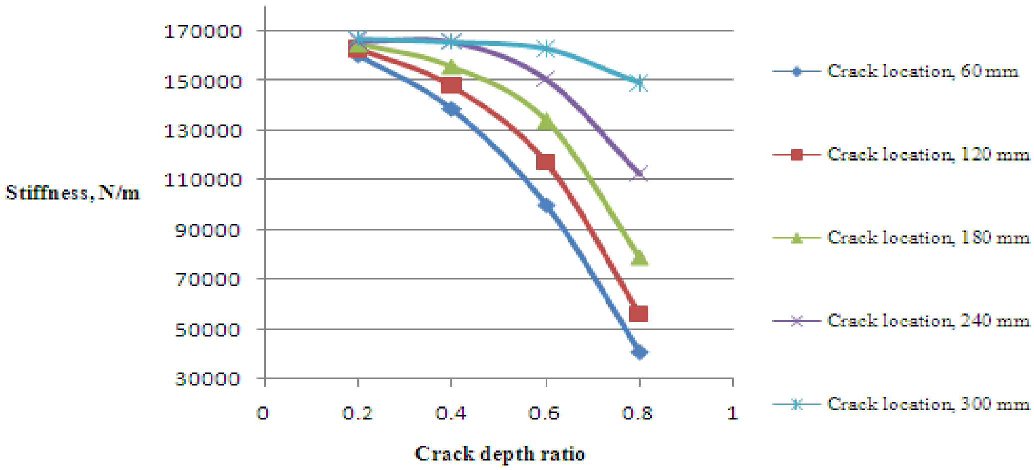

Figure-16. Variation of stiffness versus crack depth ratio for single cracked specimen.

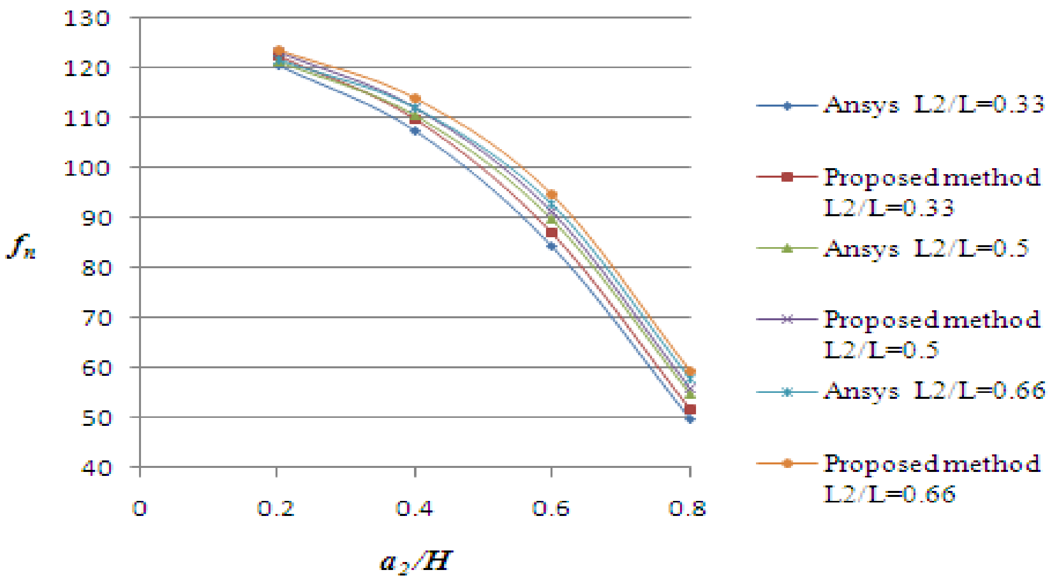

Figure-17. Variation of natural frequency versus crack depth ratio for double cracked specimen.

From Figure-15, it is also found that when the crack location increases from the fixed end at constant crack depth, then the value of stiffness increases very gradually. It is found true when crack depth is either equal to or less than 20% of the total beam depth. A constant value of stiffness means an almost constant natural frequency of the beam. For larger crack depth, stiffness increases remarkably as crack location increases.

From Figure-16, it is found that as crack depth is increased at any unique location, then stiffness of the beam decreases, and hence the natural frequency. Therefore, it is clear that change in crack depth is a function of stiffness. The stiffness of the beam decreases considerably when crack depth increases to 80% of the beam depth, and it is least affected when crack depth is either 20% or less than 20% of the total depth the beam. It is also found that for 60 mm crack location, when crack depth is increased, then stiffness decreases more abruptly than 120 mm, 180 mm, 240 mm and 360 mm crack location. This is because of most damping effect at 60 mm crack location.

From Figure-17, it is found that when the depth of second crack increases, then natural frequency decreases, this is true for all the cracked cases of a beam. The reduction in the natural frequency is only due to the increased removal of material from the beam. From Figure-17, it is also found that the reduction in natural frequency is least abrupt when second crack location (L_2/L) is equal to 0.66. It means that when the second crack location remains farthest away from the cantilever end; then least effect of damping get present in the beam than all the cracked cases in which the second crack location is comparatively closer to the fixed end. It is also observe that the results given by the proposed theoretical method for the natural frequency are extremely outstanding and can be seen from Figure-17.

4. CONCLUSIONS

- The converge formula of a proposed theoretical method of an un-cracked beam can be extended to single edge or double edge cracked cantilever beam as it gives good results for natural frequency.
- The proposed theoretical method gives one more and significant way to the researchers for evaluating natural frequency of a cracked cantilever beam.
- Natural frequency obtained by the proposed theoretical method and ANSYS software gives good agreements; it shows authenticity of the proposed method.
- Zero frequency deflection approach used in ANSYS, gives outstanding results of stiffness for intact and for cracked cases of a cantilever beam.
- When the location of crack is kept constant and crack depth is increased, then stiffness of the beam decreases.
- When the crack depth is kept constant and crack location is varied from the cantilever end, then stiffness of the beam increases.
- For single edge cracked beam, at last location (300 mm from the fixed end), even though crack depth increases, then the value of stiffness of a beam decreases slightly as compared to 60 mm, 120 mm, 180 mm, and 240 mm location.
- The effect of second crack location remains least on the natural frequency when second crack location remains nearer to the free end of a cantilever beam.

NOMENCLATURES

a_1 = First crack depth, mm	K = *Stiffness* of intact or cracked beam, N/m
a_2 = Second crack depth, mm	L = Length of beam, m
A = Cross sectional area of beam, m^2	L_1 = Location of the first crack from the fixed end, mm
B = Breadth of beam, m	L_2 = Location of the second crack from the fixed end, mm
H = Height or depth of beam, m	m = Mass of a cantilever beam, Kg
E = Young's modulus of beam, N/m^2	*Greek Symbols*
f_n = Natural Frequency, Hz	μ = Poisson's ratio
I = Moment of inertia of beam, m^4	ρ = Density of beam, Kg/m^3

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