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DYNAMICS OF A TIME DELAYED ECOLOGICAL MODEL COMPRISING MUTUALISM, NEUTRALISM AND PREY-PREDATION

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ABSTRACT

The present paper deals with the stability analysis of a three species ecological model with mutualism, neutralism and prey-predation. Here the second species is a predator preying on the first species and the third species is mutual to the first and neutral to the second species. In addition to the interaction among the species which helps in their respective growth rates, the species are provided with an alternate food resource. Also a more symmetric form of time delay is assumed in the interaction between the predator and the prey species. The model is governed by a system of integro-differential equations. Using Routh-Hurwitz criterion the local stability is established at the interior equilibrium point and a Lyapunov function is constructed to study the global asymptotic stability of the system. Some valid conclusions regarding the sustainability of the system are made at the end using numerical simulation.

Keywords: mutualism, neutralism, prey-predation, time delay, global stability, AMS subject classification: 92D40, 34Dxx.

1. INTRODUCTION

For the past few decades, Mathematical modeling has been captivating several mathematicians and ecologists. The branch of mathematical ecology showed a significant insight into the complicated biological processes. At the beginning, several two species ecological models with interactions primarily on prey-predation were studied by well known ecologists like Lotka [1] and Volterra [2]. However, the later years showed a significant development in the ecology field with the contribution of famous authors like Kapur [3], [4], Freedman [5], Paul Colinvaux [6], Cushing [7]. The local and global stability analysis of the ecological model mainly describes the dynamical behavior of the system. Several biological factors, maturation times, seasonal variations, gestation periods, atmospheric components which influenced significant change in the strength of the species permitted and created enthusiasm in incorporating delay arguments in these models. Recently various authors like Lakshmi Narayan [8], Paparao [9], Ranjith [10], Vidyanath [11] Studied and analyzed different interaction models with discrete and continuous time delay of two and three species applying new techniques. Popular researchers like Sreehari Rao [12], Gopala Swamy [13] and Yang Kuang [14] have extensively discussed the dynamics and control of biological systems with time delays. With this brief description, we have considered a three species ecological model with mutualism, neutralism and prey-predation. The second species is a predator preying on the first and the third species is mutual to the first and neutral to the second. Also we have incorporated a distributed time delay in the interaction between the prey and the predator. In general, the distributed time delays are more practical and sometimes become difficult to simplify with and the delay kernel k may be complex to approximate from the known data. The delay models establish the survival of the system at each stage and help us understand the biological phenomena of the system. Here we tried to analyze the new dynamics of the model besides their oscillatory existence by introducing these time delays. The model is characterized by a system of integro-differential equations. The local stability is derived at the interior equilibrium point and the Global stability of the system is established by constructing a suitable Lyapunov function. Finally with the help of Matlab numerical simulation is performed by identifying a suitable set of parametric values.

The paper is organized as follows.

The proposed model is defined in the next section and its interior equilibrium point is obtained in section 3. The local stability of the interior equilibrium point is established in section 4 and in section 5 the global asymptotic stability of the system is derived by the construction of a suitable Lyapunov function. With the help of Matlab software, numerical simulation is performed in section 6 by assuming suitable parameter values. Some concluding remarks based on the dynamics of the system were made in the 7th section.

2. THE MODEL

The model is described by the following system of integro-differential equations:

(i) The growth rate of first species (N_1):

$$\frac{dN_1}{dt} = a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1\int_{-\infty}^{T} k_2(t-s)N_2(s)ds + \alpha_{13}N_1N_3$$

(ii) The growth rate of second species (N_2):

$$\frac{dN_2}{dt} = a_2N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_2\int_{-\infty}^{T} k_1(t-s)N_1(s)ds \qquad (1)$$

(iii) The growth rate of third species (N_3):

VOL. 12, NO. 22, NOVEMBER 2017 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

$$\frac{dN_3}{dt} = a_3 N_3 - \alpha_{33} N_3^2 + \alpha_{31} N_3 N_1$$

N_i's are the population densities and a_i's are the respective intrinsic growth rates of all the three species for i=1, 2, 3. α_{13} and α_{31} are the rates of increase of N_1 and N_3 respectively due to their mutuality nature. α_{12} is the rate of decrease of N_1 due to successful attacks by the predator N_2 and α_{21} is the rate of increase of N_2 due to interaction with N_1. α_{ii} 's (i=1,2,3) are the natural death rates of all the three species due to inter competitions.

Also $k_1(t-s), k_2(t-s)$ presents the weight factors which impacts the population sizes of N_1 & N_2 after a time interval $(t-s)$ for all $t \geq s$.
Let $t - s = z \Rightarrow s = t - z$.

Hence, $k_1(z), k_2(z) \geq 0$ and by normalizing so that $\int_0^\infty k_1(z)dz = \int_0^\infty k_2(z)dz = 1$ (2)

Therefore, the proposed system is rewritten after applying the delay kernel conditions:

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 \int_0^\infty k_2(z) N_2(t-z)dz + \alpha_{13} N_1 N_3$$

$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_2 \int_0^\infty k_1(z) N_1(t-z)dz \quad (3)$$

$$\frac{dN_3}{dt} = a_3 N_3 - \alpha_{33} N_3^2 + \alpha_{31} N_3 N_1$$

If the delay kernel is identically equal to zero for all $t \geq s$ then the delay represented by the integral form in the above equation is called bounded delay otherwise it becomes unbounded.

3. THE POSITIVE EQUILIBRIUM STATE

The state in which all the three species exist

$$\overline{N_1} = \frac{\alpha_{33}(a_1\alpha_{22} - a_2\alpha_{12}) + a_3\alpha_{13}\alpha_{22}}{\alpha_{22}(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}) + \alpha_{12}\alpha_{21}\alpha_{33}}$$

$$\overline{N_2} = \frac{a_1\alpha_{21}\alpha_{33} + a_2(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}) + a_3\alpha_{21}\alpha_{13}}{\alpha_{22}(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}) + \alpha_{12}\alpha_{21}\alpha_{33}}$$

$$\overline{N_3} = \frac{\alpha_{31}(a_1\alpha_{22} - a_2\alpha_{12}) + a_3(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})}{\alpha_{22}(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}) + \alpha_{12}\alpha_{21}\alpha_{33}}$$

The equilibrium state exist only when,

$$\alpha_{11}\alpha_{33} > \alpha_{13}\alpha_{31} \;\&\; a_1\alpha_{22} > a_2\alpha_{12}$$

4. THE LOCAL STABILITY OF THE POSITIVE EQUILIBRIUM STATE

Let $N = (N_1, N_2, N_3)^T = \overline{N} + U$

Where U = $(u_1, u_2, u_3)^T$ is the perturbation over the equilibrium state. $\overline{N} = (\overline{N_1}, \overline{N_2}, \overline{N_3})^T$ .The basic equations (2.3) are linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt} = AU \quad (4)$$

Where

$$A = \begin{bmatrix} a_1 - 2\alpha_{11}N_1 - \alpha_{12}N_2 + \alpha_{13}N_3 & -\alpha_{12}N_1k_2^*(\lambda) & \alpha_{13}N_1 \\ \alpha_{21}N_2k_1^*(\lambda) & a_2 - 2\alpha_{22}N_2 + \alpha_{21}N_1 & 0 \\ \alpha_{31}N_3 & 0 & a_3 - 2\alpha_{33}N_3 + \alpha_{31}N_1 \end{bmatrix}$$

The characteristic equation for the system is $det\left[A - \lambda I\right] = 0$ (5)

The equilibrium state is stable, if the roots of the equation (5) are negative in case they are real or the roots have negative real parts in case they are complex.
The linearized equations are

$$\frac{du_1}{dt} = -\alpha_{11}\overline{N_1}u_1 - \alpha_{12}\overline{N_1}k_2^*(\lambda)u_2 + \alpha_{13}\overline{N_1}u_3$$

$$\frac{du_2}{dt} = \alpha_{21}\overline{N_2}k_1^*(\lambda)u_1 - \alpha_{22}\overline{N_2}u_2 \quad (6)$$

$$\frac{du_3}{dt} = \alpha_{31}\overline{N_3}u_1 - \alpha_{33}\overline{N_3}u_3$$

With the characteristic equation

$$\lambda^3 + \left(\alpha_{11}\overline{N_1} + \alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3}\right)\lambda^2$$
$$+ \left[\left(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}k_1^*(\lambda)k_2^*(\lambda)\right)\overline{N_1}\,\overline{N_2} + \left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right)\overline{N_1}\,\overline{N_3} + \alpha_{22}\alpha_{33}\overline{N_2}\,\overline{N_3}\right]\lambda$$
$$+ \left[\alpha_{22}\left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right) + \alpha_{12}\alpha_{21}\alpha_{33}k_1^*(\lambda)k_2^*(\lambda)\right]\overline{N_1}\,\overline{N_2}\,\overline{N_3} = 0$$

Let

VOL. 12, NO. 22, NOVEMBER 2017 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

$$h_1 = \alpha_{11}\overline{N_1} + \alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3}$$

$$h_2 = \left(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}k_1^*(\lambda)k_2^*(\lambda)\right)\overline{N_1}\,\overline{N_2} + \left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right)\overline{N_1}\,\overline{N_3} + \alpha_{22}\alpha_{33}\overline{N_2}\,\overline{N_3}$$

$$h_3 = \left[\alpha_{22}\left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right) + \alpha_{12}\alpha_{21}\alpha_{33}k_1^*(\lambda)k_2^*(\lambda)\right]\overline{N_1}\,\overline{N_2}\,\overline{N_3}$$

By Routh-Hurwitz criteria, the system is stable if $D_1 = h_1 > 0$, $D_2 = \left(h_1h_2 - h_3\right) > 0$ and $D_3 = h_3\left(h_1h_2 - h_3\right) > 0$

Clearly $h_1 > 0$ and also by certain algebraic deductions we have,

$$\begin{aligned} D_2 &= h_1h_2 - h_3 \\ &= \left(\alpha_{11}\overline{N_1} + \alpha_{22}\overline{N_2}\right)\left(\alpha_{11}\alpha_{22}\overline{N_1}\,\overline{N_2} + \alpha_{12}\alpha_{21}\overline{N_1}\,\overline{N_2}k_1^*(\lambda)k_2^*(\lambda)\right) \\ &\quad + \left(\alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3}\right)\alpha_{22}\alpha_{33}\overline{N_2}\,\overline{N_3} + 2\alpha_{11}\alpha_{22}\alpha_{33}\overline{N_1}\,\overline{N_2}\,\overline{N_3} \\ &\quad + \left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right)\left(\alpha_{11}\overline{N_1}^2\overline{N_3} + \alpha_{33}\overline{N_1}\,\overline{N_3}^2\right) \\ &> 0 \end{aligned}$$

Which is possible if $\alpha_{11}\alpha_{33} > \alpha_{13}\alpha_{31}$

Therefore, the positive equilibrium point is locally asymptotically stable.

5. GLOBAL STABILITY

The following Lyapunov function is chosen for the interior equilibrium point:

$$V(\overline{N_1},\overline{N_2},\overline{N_3}) = \sum_{i=1}^{3} N_i - \overline{N_i} - \overline{N_i}\ln\left(\frac{N_i}{\overline{N_i}}\right) + \frac{1}{2}\alpha_{12}\int_0^\infty k_2^*(z)\int_{t-z}^{t}\left[N_2 - \overline{N_2}\right]^2 du\,dz + \frac{1}{2}\alpha_{21}\int_0^\infty k_1^*(z)\int_{t-z}^{t}\left[N_1 - \overline{N_1}\right]^2 du\,dz \quad (7)$$

Then calculate $\dfrac{dV}{dt}$ which is as follows

$$\frac{dV}{dt} = \sum_{i=1}^{3}\left(N_i - \overline{N_i}\right)\frac{1}{N_i}\frac{dN_i}{dt} + \frac{1}{2}\alpha_{12}\int_0^\infty k_2^*(z)\left[N_2 - \overline{N_2}\right]^2 dz - \frac{1}{2}\alpha_{12}\int_0^\infty k_2^*(z)\left[N_2(t-z) - \overline{N_2}\right]^2 dz + \frac{1}{2}\alpha_{21}\int_0^\infty k_1^*(z)\left[N_1 - \overline{N_1}\right]^2 dz - \frac{1}{2}\alpha_{21}\int_0^\infty k_1^*(z)\left[N_1(t-z) - \overline{N_1}\right]^2 dz$$

Using the model equations (1) and the normalizing conditions (2) we get,

$$\frac{dV}{dt} = \left[N_1 - \overline{N_1}\right]\left[a_1 - \alpha_{11}N_1 - \alpha_{12}\int_0^\infty k_2^*(z)N_2(t-z)\,dz + \alpha_{13}N_3\right] + \left[N_2 - \overline{N_2}\right]\left[a_2 - \alpha_{22}N_2 + \alpha_{21}\int_0^\infty k_1^*(z)N_1(t-z)\,dz\right] + \left[N_3 - \overline{N_3}\right]\left[a_3 - \alpha_{33}N_3 + \alpha_{31}N_1\right]$$

Choosing,

$$a_1 = \alpha_{11}\overline{N}_1 + \alpha_{12}\int_0^\infty k_2^*(z)N_2(t-z) - \alpha_{13}\overline{N_3},$$

$$a_2 = \alpha_{22}\overline{N_2} - \alpha_{21}\int_0^\infty k_1^*(z)N_1(t-z)\,dz$$

$$a_3 = \alpha_{33}\overline{N}_3 - \alpha_{31}\overline{N}_1$$

We get

$$\frac{dV}{dt} = \left(-\alpha_{11} + \frac{1}{2}\alpha_{21}\right)\left[N_1 - \overline{N_1}\right]^2 + \left(-\alpha_{22} + \frac{1}{2}\alpha_{12}\right)\left[N_2 - \overline{N_2}\right]^2 + \left(\alpha_{13} + \alpha_{31}\right)\left[N_1 - \overline{N_1}\right]\left[N_3 - \overline{N_3}\right] - \alpha_{33}\left[N_3 - \overline{N_3}\right]^2 - \frac{1}{2}\alpha_{12}\int_0^\infty k_2(z)\left[N_2(t-z) - \overline{N_2}\right]^2 dz - \frac{1}{2}\alpha_{21}\int_0^\infty k_1(z)\left[N_1(t-z) - \overline{N_1}\right]^2 dz$$

Using the inequality, $ab \le \dfrac{a^2 + b^2}{2}$ *and*

$$\int_0^\infty k_2(z)\left[N_2(t-z) - \overline{N_2}\right]^2 dz \le \int_0^\infty k_2(z)\,dz = 1$$

$$\int_0^\infty k_1(z)\left[N_1(t-z) - \overline{N_1}\right]^2 dz \le \int_0^\infty k_1(z)\,dz = 1$$

we have

$$\begin{aligned} \frac{dv}{dt} &\le -\left[\alpha_{11} - \frac{1}{2}\left(\alpha_{21} + \alpha_{13} + \alpha_{31}\right)\right]\left[N_1 - \overline{N_1}\right]^2 - \left[\alpha_{22} - \frac{1}{2}\alpha_{12}\right]\left[N_2 - \overline{N_2}\right]^2 \\ &\quad - \left[\alpha_{33} - \frac{1}{2}\left(\alpha_{13} + \alpha_{13}\right)\right]\left[N_3 - \overline{N_3}\right]^2 - \frac{1}{2}\left(\alpha_{12} + \alpha_{21}\right) \\ &\le 0 \end{aligned} \quad (8)$$

Hence the system is globally asymptotically stable at the interior equilibrium point.

6. NUMERICAL EXAMPLES

Case(i): a_1=5; α_{11}=0.05; α_{12}=0.15; α_{13}=0.15; a_2=0.25; α_{21}=0.25; α_{22}=0.015; a_3=2.5; α_{31}=0.05; α_{33}=0.25;

VOL. 12, NO. 22, NOVEMBER 2017 ISSN 1819-6608
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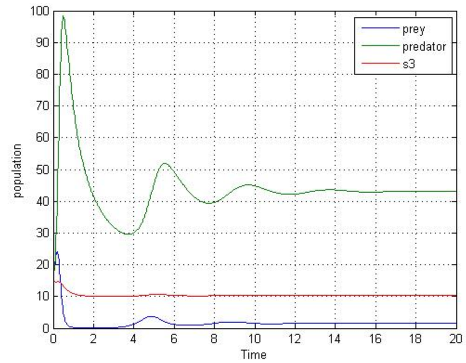
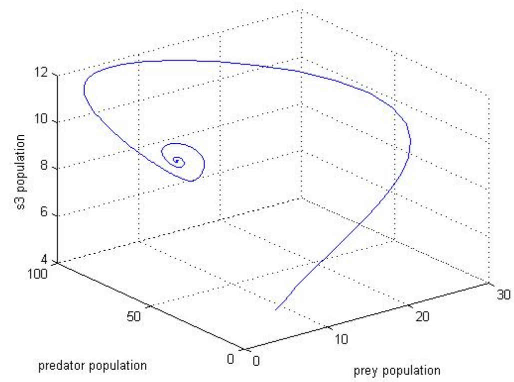
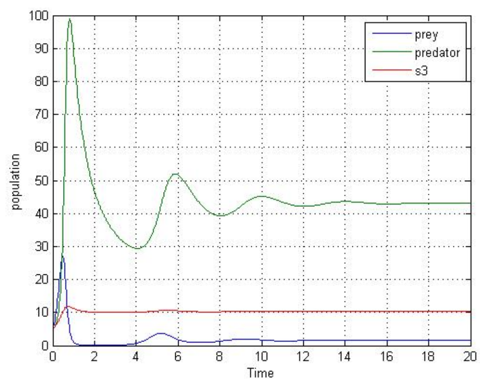
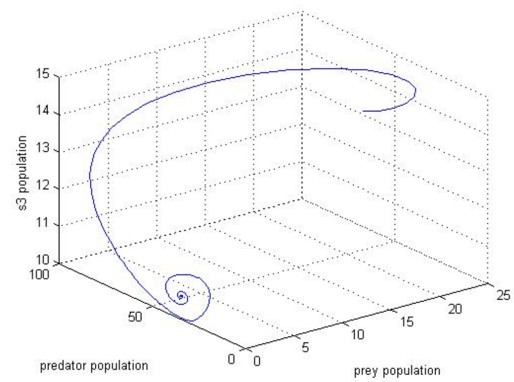




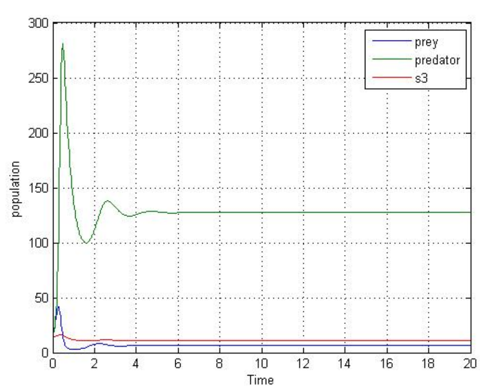


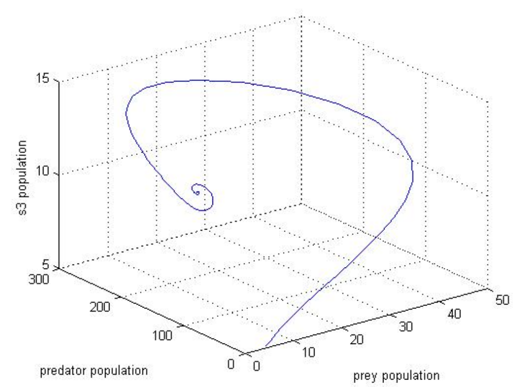


With the assumed parameter values satisfying the equilibrium point existence conditions and the initial strengths of the populations as 5, 5, 5 respectively the system becomes asymptotically stable and reaches the equilibrium point (1.5, 43.1, 10.3) as the time increases.

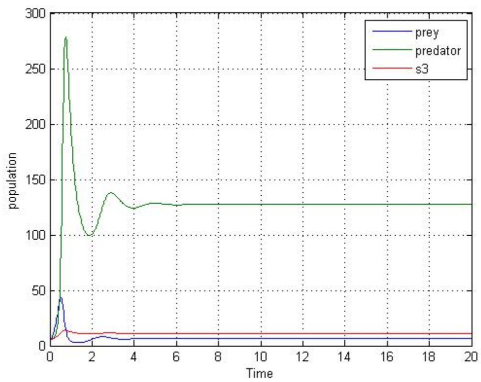
An increase in the strength of the populations to 15, 15, 15, the system remains stable and converges to the equilibrium point (1.6, 43.2, 10.3). The time series graph and also the phase portrait graph shows the global asymptotic stability of the system. In both cases the predator population increases rapidly in the beginning and falls to a certain level leading to a stable nature.

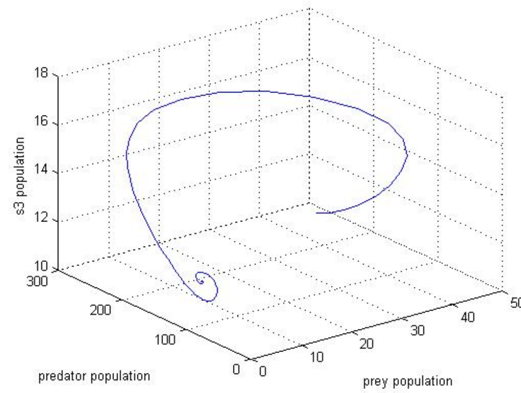
Case(ii): a_1=5; α_{11}=0.05; α_{12}=0.15; α_{13}=0.15; a_2=0.3; α_{21}=0.25; α_{22}=0.015; a_3=2.5; α_{31}=0.05; α_{33}=0.25;





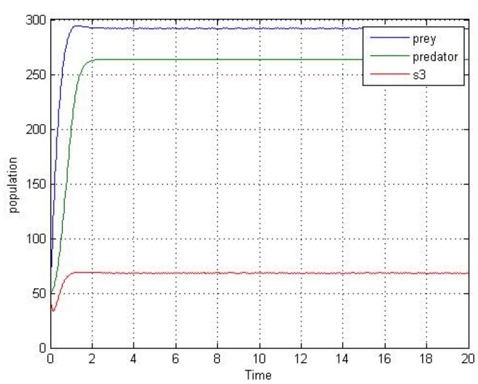
VOL. 12, NO. 22, NOVEMBER 2017 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

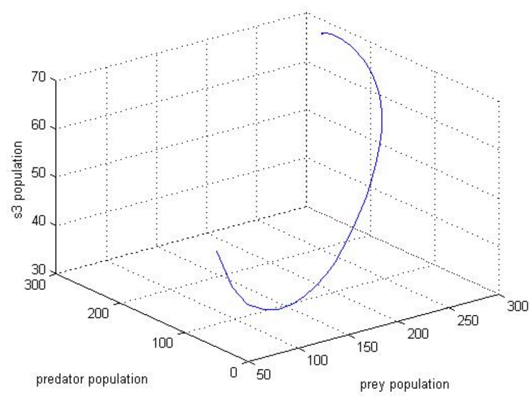




A slight variation in the intrinsic growth rate of the predator population is shown here without delay kernels. When all the other parameter values remain the same, the predator population is further increasing rapidly and falling down in a similar fashion which leads to the stable nature of the system. The time series graph and phase space portrait shows the these dynamics and the system reaches the equilibrium point (6.4, 127.4, 11.3) with increase in time.

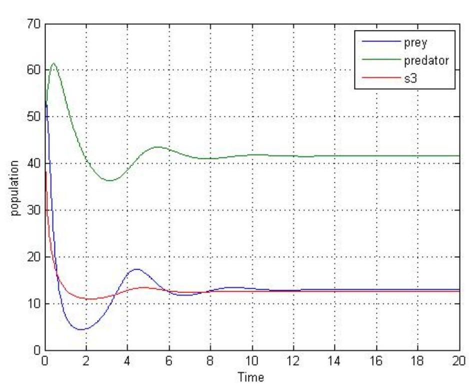
Case(iii): a_1=5; α_{11}=0.05; α_{12}=0.15; α_{13}=0.15; a_2=0.3; α_{21}=0.25; α_{22}=0.015; a_3=2.5; α_{31}=0.05; α_{33}=0.25;

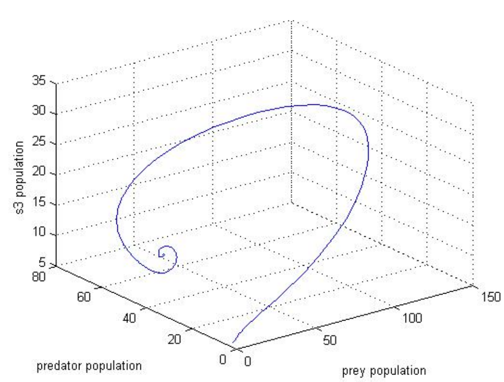




To study the dynamics of the system with equal delay kernels α=20 & β=20, and increasing the strength of the species to 50, 50, 50, we observe that the strength of the prey and the predator species increasing rapidly as the time increases and the system seems to be stabilizing to the equilibrium point (292.3, 263.4, 68.4).

Case(iv): a_1=5; α_{11}=0.05; α_{12}=0.15; α_{13}=0.15; a_2=0.3; α_{21}=0.25; α_{22}=0.015; a_3=2.5; α_{31}=0.05; α_{33}=0.25;





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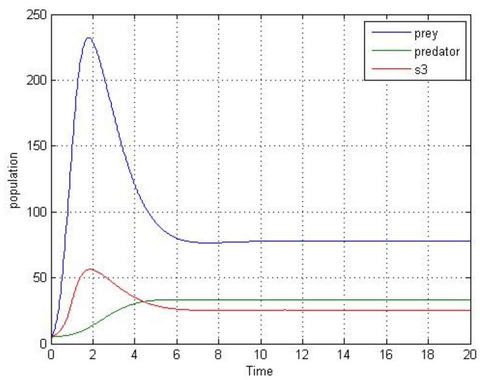
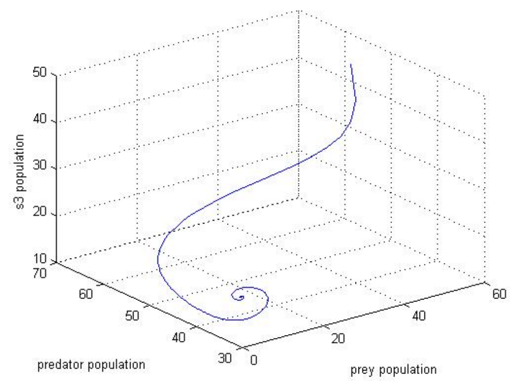
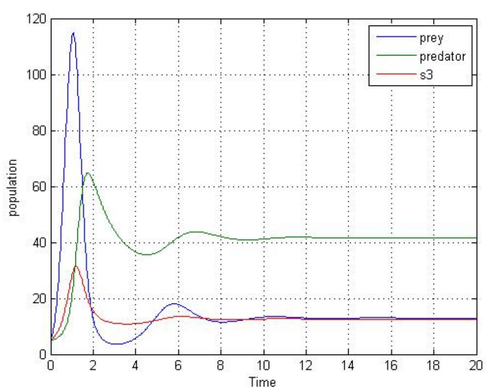
Now, keeping β fixed and changing α to 10, we observe that the prey population falls down rapidly and then stabilizes slowly. Also the predator gains its strength initially and as the time increases the system reaches to the equilibrium point (12.6, 41.5, 12.5) which is shown in the graphs.

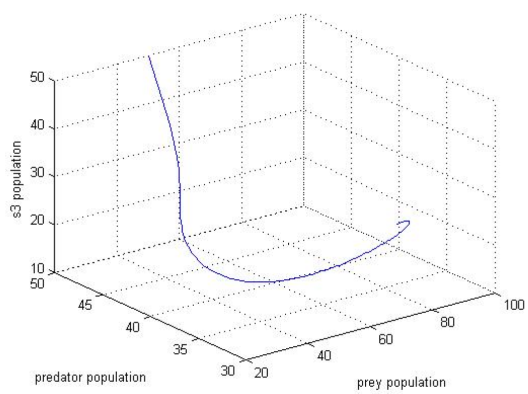


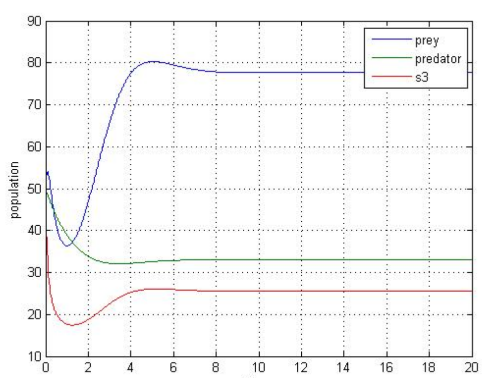


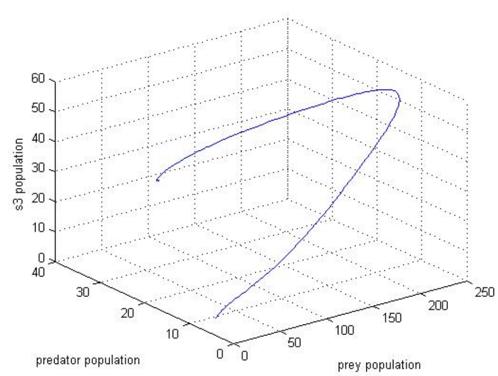
Further increase in the delay kernel α to 100 shows the asymptotic stability of the system and tends to the equilibrium point (77.8, 33, 25.6). We also observe a significant growth rate in the prey species and decrease in the population of the predator species.

Keeping β fixed again and taking the delay kernel α=10, a significant growth rate is observed again in all the three species and further the system stabilizes with increase in time and it reaches to the equilibrium state (12.6, 41.6, 12.5).







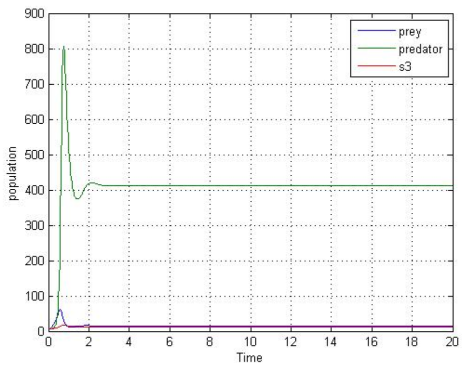


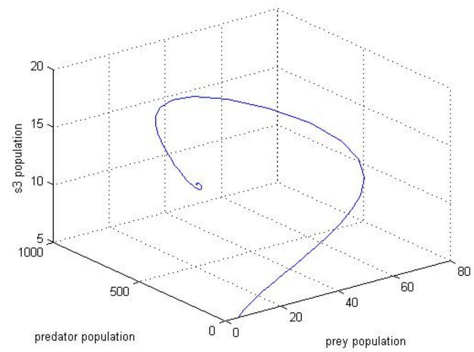
Increase in the delay kernel α to 100, the strengths of all the three species decreases initially and as time increases a significant growth is observed in the prey population which further stabilizes the system to the

VOL. 12, NO. 22, NOVEMBER 2017 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

equilibrium point (77.8, 33, 25.5). The times series analysis and the phase space trajectory shows the stability nature of the system.

Case(v): a_1=5; α_{11}=0.05; α_{12}=0.15; α_{13}=0.15; a_2=0.3; α_{21}=0.25; α_{22}=0.015; a_3=2.5; α_{31}=0.05; α_{33}=0.25;





The above graphs show the variations of all the three species with respect to time again when the delay kernel α is fixed. A significant increase in the delay kernel β to 10 leads to rapid growth in the predator population initially for some time period and since prey species is effected much by this delay it is clear from the graph that the limited availability of the prey species leads the system to the equilibrium state (22.9, 402, 14.6). Also we observe that the prey species may become extinct if the delay kernel β is increased further which destabilizes the system.

7. CONCLUSIONS

A three species ecological model with mutualism, neutralism and prey-predation is taken up for stability analysis in this paper. The model is characterized by a system of integro-differential equations in which we have imposed delay in the interaction between prey and predator species A more symmetric form of modified prey-predator model with time delay studied by Volterra is employed here. The local stability of the system is derived and the global stability is established with the help of Lyapunov function. The numerical simulations with the help of Matlab satisfying the conditions for the existence of the interior equilibrium point shows that the system is globally asymptotically stable. Also we observe that the delay kernels effect the stability and behavior of the system. Primarily we observe that the increase in the delay kernel α leads to a significant growth in the prey population and further helps in stabilizing the system. But a significant increase in the delay kernel β leads to rapid growth in the predator population which may effect the stability of the system as there are more chances for the prey species to become extinct.

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