DIFFUSION ANALYSIS OF A PREY PREDATOR FISHERY MODEL WITH HARVESTING OF PREY SPECIES

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ABSTRACT
In this research article, we considered an ecological prey predator fishery model system with a generalized case where both the patches are accessible to both prey and predator. We suppose that the prey migrate between two patches randomly. The growth of prey in each patch in the absence of predators is assumed to be logistic. The predator consumes the prey with intrinsic growth rates in both the patches. The existence of its steady states and their stability (local and global) are analyzed. It is also emphasized the diffusive stability of the system along with some numerical simulations. Numerical simulation has also been performed in support of analysis by using MATLAB.

Keywords: prey predator, local and global stability, boundedness, harvesting, bionomic equilibrium, diffusion.

1. INTRODUCTION
There is a need to protect the fish population, major food recourse, by restricting exploitation of the fish, creating natural resources and creating of protected zones for them. By these measures the species will grow without any disturbances and hence protected population can improve their numbers. Mathematical modeling is an important tool which involves the study of various disciplines like Genetics, pharmacokinetics, epidemiology, ecology, biology etc. In recent years, this modeling is raising and spreading to branches of life sciences. Bio Mathematics has been useful in recent years to various problems in ecology and epidemiology [1, 2]. Policymakers and Scientists who are working in marine fisheries and others discussed marine protected areas by considering the economic and social benefits [3]. Marine reserve may be introduced as a defending measure because it is assumed that adult or juvenile migration will replenish depleted fishing grounds beyond the borders of the marine reserve. The benefits of creating marine reserves can go beyond the safeguard of a specific overfished fish population. Marine reserve can guard the marine landscape from degradation caused by damaging fishing practices; provide an important prospect to learn about marine ecosystems. It helps to investigate on species dynamics and management tools on protection of all components of a marine community. Marine reserves have provided many benefits as a tool for conservation and marine environmental management [4]. Takashina et al. [5] investigated the prospective influence of starting of marine sheltered zones on aqua ecosystems using different mathematical tools and validated that founding of marine sheltered zone can result in a substantial deterioration of species. Wang and Takeuchi [6] suggested tools to simulate movements of species between zones. They have revealed that there exists an arrangement that species collabate well through adaptations such that predators become extinct in each patch in the absence of adaptations. Dubey [7] suggested a mathematical structure to study the impact of a reserved zone on the dynamics of an aqua eco system. Many of the reimbursements accompanying with marine sheltered zones have been extensively explored and the arena is an active area of research in theoretical ecology and mathematical biology [8]. The authors [9, 10] investigated the stability of three species and four species with migration, bionomic equilibrium and optimal harvesting policy. The present investigation is a study of diffusion stability of a prey-predator system with logistic growth in a two zone aqua environment.[11-17] inspired us to consider to do the present investigation is on the analytical and numerical approach of diffusive stability of an aqua eco system.

2. MATHEMATICAL EQUATIONS
Consider a marine environment (Figure-1) where prey and predator species living together with the following assumptions: (1) It is a prey-predator system in a two patch environment (2) Both are accessible to both prey and predators. (3) Each patch is supposed to be homogeneous (4) We suppose that the prey migrate between the two patches randomly (5) The growth of prey in each patch in the absence of predators is assumed to be logistic (6) The predator consumes the prey in the zone and grows logistically with intrinsic growth rates $a_1$ and $a_2$ carrying capacities proportional to the population size.

![Figure-1. Marine environment.](image-url)
Let \( x(t), y(t), E_1, K, \gamma_1, m_1, q_1, r, \alpha_1 \) represent biomass density of prey species, biomass density of the predator species, the effort applied to harvest the fish population, carrying capacity of prey species, the equilibrium ratio of prey to predator biomass, the mortality rate due to predation, catch ability coefficient, intrinsic growth rate of prey species, intrinsic growth rates of predators, respectively in patch-1.

Let \( y(t), w(t), E_2, L, \gamma_2, m_2, q_2, s, \alpha_2 \) represents biomass density of prey species, biomass density of the predator species, the effort applied to harvest the fish population, carrying capacity of prey species, the equilibrium ratio of prey to predator biomass, the mortality rate due to predation, catchability coefficient, intrinsic growth rate of prey species, intrinsic growth rates of predators respectively, in patch-2. Let \( \sigma_1, \sigma_2 \) represents the migration rates from patch-1 to patch-2 and vice-versa. Let \( x = x(u,t) \) represents the biomass density of prey in patch-1, \( y = y(u,t) \) represents the biomass density of prey in patch-2, \( z = z(u,t) \) represents the biomass density of predator in patch-1, \( w = w(u,t) \) represents the biomass density of predator in patch-2. \( D_1, D_2, D_3, D_4 \) represents the diffusion coefficients of prey and predator species in patch-1 and patch-2 respectively. Keeping these in view, the dynamics of the system may be governed by the following partial differential equations

\[
x_t = rx(1-(x/K)) - \sigma_1x + \sigma_2 y - m_1 x z - q_1 E_1 x + D_1 x_{uu} \\
y_t = sy(1-(y/L)) + \sigma_2x - m_2 y w - q_2 E_2 y + D_2 y_{uu} \\
z_{tt} = \alpha_1 z (1-(z/\gamma_1 x)) + D_3 z_{uu} \\
w_{tt} = \alpha_2 w (1-(w/\gamma_2 y)) + D_4 w_{uu}
\]

where \( x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0 \)

Throughout this analysis we assume that

\[
r - \sigma_1 - q_1 E_1 > 0, \quad s - \sigma_2 - q_2 E_2 > 0
\]

3. EXISTENCE OF EQUILIBRIA AND STABILITY IN THE ABSENCE OF DIFFUSION

Steady states of equations 1-4 are given by (i) \( P_1(0,0,0,0) \) (ii) \( P_2(x^*, y^*, 0, 0) \) (iii) \( P_3(x^*, y^*, 0, 0) \) (iv) \( P_4(x^*, y^*, z^*, w^*) \). Since we are interesting to study the assumed ecological system about interior steady state, let \( x^*, y^*, z^*, w^* \) are positive solutions of

\[
x'(t) = 0, y'(t) = 0, z'(t) = 0 \text{ and } w'(t) = 0, \text{ where}
\]

\[
y^* = \frac{1}{\sigma_2} \left( \frac{m_1 \gamma_1 + (r/K)(x^*)^2}{q_1 E_1 + \sigma_1 - \gamma_1 x^*} \right), \quad z^* = \gamma_1 x^*,
\]

\[
w^* = \gamma_2 y^* \text{ and } x^* \text{ is the solution of}
\]

\[
a_3x^3 + b_3(x^2)^2 + c_3x^2 + d_3 = 0
\]

where\[
\begin{align*}
a_3 &= \frac{1}{\sigma_2} \left[ (\frac{(s/L) + m_2 \gamma_2}{m_1 \gamma_1 + (r/K)})^2 \right] \\
b_3 &= (-2) \left[ (\frac{(s/L) + m_1 \gamma_1}{m_1 \gamma_1 + (r/K)})(r - \gamma_1 q_1 E_1) \right] / \sigma_2^2 \\
c_3 &= \frac{1}{\sigma_2^2} \left( \frac{(s/L) + m_2 \gamma_2}{m_1 \gamma_1 + (r/K)} \right)^2 \\
d_3 &= \left( \frac{r - \gamma_1 q_1 E_1}{s - \gamma_2 q_2 E_2} \right)(s - \gamma_2 q_2 E_2) / \sigma_2^2 - \gamma_1
\end{align*}
\]

Equation has a unique positive solution if

\[
\frac{1}{\sigma_2^2} \left( \frac{\gamma_1}{L} + m_2 \gamma_2 \right) / (s - \gamma_2 q_2 E_2) < (s - \gamma_2 q_2 E_2) \left( \frac{m_1 \gamma_1 + \gamma_1}{r/K} \right) < \frac{1}{\sigma_2^2} \left( \frac{\gamma_1}{L} + m_2 \gamma_2 \right) / (s - \gamma_2 q_2 E_2)
\]

For \( y^* \) to be positive, we must have

\[
(s - \gamma_2 q_2 E_2) / (r - \gamma_1 q_1 E_1) < \sigma_2 \gamma_1
\]

For \( x^* \) to be positive, we must have

\[
(s - \gamma_2 q_2 E_2) / (r - \gamma_1 q_1 E_1) < (s/L) + m_2 \gamma_2
\]

Local stability: Let us now suppose that the above system has a unique positive equilibrium at \( P_3(x^*, y^*, z^*, w^*) \) and the dynamics of the Jacobian matrix of the system at \( P_3(x^*, y^*, z^*, w^*) \) is given by

\[
\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0
\]

where \( a_1 = a + b + \alpha_1 + \alpha_2 \)

\[
a_2 = ab + ba_1 + \alpha_1 \alpha_2 + ba_2 + ab \alpha_1 + \alpha_2 \alpha_1 + \alpha_2 \alpha_2 \\
+m_2 \alpha_2 \gamma_2 y^* - \sigma_1 \sigma_2 + m_1 \alpha_1 \gamma_1 x^* \\
a_3 = a_1 \alpha_1 \alpha_2 + ba_1 \alpha_2 + ab \alpha_1 + ba \alpha_2 \\
+am \alpha_2 \gamma_2 y^* + a_1 \alpha_1 m_2 \gamma_2 y^* - \alpha_1 \gamma_1 \sigma_2 - \alpha_2 \gamma_2 \sigma_2 \\
+m_2 x \alpha_1 \alpha_2 \gamma_1 + m_1 x^* ba \alpha_1 \gamma_1 \\
a_4 = ab \alpha_1 \alpha_2 + ab \alpha_1 \alpha_2 \gamma_1 - \alpha_1 \gamma_1 \sigma_2 \sigma_2
\]
are and time variable \( \gamma \) as given by
\[ \beta \alpha \eta \gamma \chi * + \eta \eta \eta \eta \gamma \chi * y * y * \]
\[ a = \frac{\alpha^x}{K} + \frac{\sigma^y}{x} \quad ; \quad b = \frac{\gamma^x}{L} + \frac{\sigma^y}{y} \]

By Routh-Hurwitz criteria, the necessary and sufficient conditions for local stability of equilibrium point \( P_t(x', y', z', w') \) are \( a_1 > 0 \), \( a_3 > 0 \), \( a_{44} > 0 \), \( a_{13}(a_{12} - a_{11}) > a_{12} \), and \( a_{11}(a_{12} - a_{11}) > a_{12} \). It is evident that \( a_1 > 0 \), \( a_3 > 0 \) and \( a_{44} > 0 \). Clearly the last two Routh-Hurwitz conditions are same. It is easily established the conditions.

**Global stability: **Now we shall discuss the global stability of the interior equilibriums, \( P_t(x', y', z', w') \) of the system 1-4 without diffusion

Theorem: If \( A < x < B \) and \( C < y < D \), where
\[ A = \frac{1}{m_1} \left[ 1 + \frac{2r}{L \gamma_1 m_1} - \frac{4r^2}{L \gamma_1 m_1 + K^2 \gamma_1^2 m_1^2} \right] > 0 \]
\[ B = \frac{1}{m_1} \left[ 1 + \frac{2r}{L \gamma_1 m_1} + \frac{4r^2}{L \gamma_1 m_1 + K^2 \gamma_1^2 m_1^2} \right] > 0 \]
\[ C = \frac{\sigma^x}{m_2 \sigma^x} \left[ 1 + \frac{2s}{m_2 \gamma_1 m_2} - \frac{4s^2}{m_2 \gamma_1 m_2 + K^2 \gamma_1^2 m_2^2} \right] > 0 \]
\[ D = \frac{\sigma^x}{m_2 \sigma^x} \left[ 1 + \frac{2s}{m_2 \gamma_1 m_2} + \frac{4s^2}{m_2 \gamma_1 m_2 + K^2 \gamma_1^2 m_2^2} \right] > 0 \]

then \( P_t(x', y', z', w') \) is globally asymptotically stable.

Proof: Let us consider Lyapunov function
\[ V(x, y, z, w) = \left[ x - x' \ln \left( \frac{x}{x'} \right) \right] + \left[ y - y' \ln \left( \frac{y}{y'} \right) \right] \\
+ \left[ z - z' \ln \left( \frac{z}{z'} \right) \right] + \left[ w - w' \ln \left( \frac{w}{w'} \right) \right] \]
\[ V'(t) = \frac{(x - x') dx}{x} + \frac{(y - y') dy}{y} + \frac{(z - z') dz}{z} + \frac{(w - w') dw}{w} \]

Choosing \( l_1 = (y') / (x') \), \( l_2 = 1 / \alpha_1 \), \( l_3 = 1 / \alpha_2 \).

\[ V'(t) = \frac{\gamma}{K} (x - x')^2 - \frac{\gamma}{L} (y - y')^2 - \frac{\gamma}{x} (x - x') (z - z') \]
\[ - \frac{\gamma}{y} (y - y') (w - w') - \frac{\gamma}{y} (x - x') (y - y')^2 \]
\[ + \frac{1}{x} (x - x') (z - z') - \frac{1}{\gamma x} (z - z')^2 \]

To prove \( V'(t) \) to be negative, we must have \( A < x < B \) and \( C < y < D \). Hence the theorem (3.1) concludes that, in the presence of predators if prey population lie in a certain interval, they may be sustained at an appropriate equilibrium level.

**4. DIFFUSION ANALYSIS**

In this section, we have inspected the steadiness of the system 1-4 in the presence of diffusion. Ecologically, it means that the movement of species at any direction for several reasons. If we assume the movement of species only in the vertical direction, then the population density variables \( x, y, z, w \) are functions of space variable \( u \) and time variable ‘\( t \)’. In this segment, we deliberated the exceptional influences of transmission of the ideal structure (1-4). \( x = x(u, t), y = y(u, t), z = z(u, t), w = w(u, t) \) where \( u \) is a space variable and \( x(u, 0) > 0, y(u, 0) > 0, z(u, 0) > 0, w(u, 0) > 0 \), for \( u \in [0, \infty] \).

The trivial fluctuation edges conditions are specified by \[ x_{u=0,R} = y_{u=0,R} = z_{u=0,R} = w_{u=0,R} = 0 \]. Now, let us consider the ideal (2.1)-(2.4) underneath trivial fluctuations edge ailments. To analyze the role of transmission on this ideal, we deliberate the linear ideal of the structure 1-4 about the interior steady state \( P_t(x', y', z', w') \) as given by

\[ x'(t) = - \frac{x}{K} x' X - m_1 x' Z + D_1 x_{uu} \]
\[ y'(t) = - \frac{y}{L} y' Y - m_2 y' W + D_2 y_{uu} \]
\[ z'(t) = D_3 z_{uu} \]
\[ w'(t) = D_4 w_{uu} \]

by putting \( x = x' + X; \quad y = y' + Y; \quad z = z' + Z; \quad w = w' + W \). Assume that the solutions of equations in the form \( X = \alpha e^{\beta u} \cos pu, Y = \alpha e^{\beta u} \cos pu, Z = \alpha e^{\beta u} \cos pu \) and

\[ W = \alpha e^{\beta u} \cos pu \]

where \( p \) is the wave numeral of perturbation, \( \lambda \) is the frequency numeral & \( \alpha_i, i = 1, 2, 3, 4 \) are the amplitudes.

\[ \mu^4 + J_1 \mu^3 + J_2 \mu^2 + J_3 \mu + J_4 = 0 \]
where $J_i = \frac{r_i x}{K} + \frac{sy_i^2}{L} + p_i^2(D_i + D_1 + D_3 + D_4)$;

$J_2 = \left( \frac{r_i x}{K} + D_3 p_2 \right) \left( \frac{sy_i^2}{L} + (D_i + D_3 + D_4)p_2 \right) + \left( \frac{sy_i p_2}{L} + D_3 p_4 \right)(D_i + D_3)p_2 + D_i D_3 p_4$;

$J_3 = \left( \frac{r_i x}{K} + D_3 p_2 \right) \left( \frac{sy_i^2}{L} + (D_i + D_3 + D_4)p_2 \right) + \left( \frac{sy_i p_2}{L} + D_3 p_4 \right)(D_i + D_3)p_2 + D_i D_3 p_4$;

$J_4 = \left( \frac{r_i x}{K} + D_3 p_2 \right) \left( \frac{sy_i^2}{L} + D_3 p_4 \right) D_i D_3 p_4$.

The following is the immediate consequence of R-H Criteria

**Theorem (1):** The point $P(x, y, z, w)$ is locally asymptotically stable in the attendance of transmission, if $J_1 > 0$, $J_2 > 0$, $J_3 > 0$, $J_4 > 0$. $J_i(J_2 - J_i) > J_i J_4$ and $J_i J_1 J_3 - J_1^2 J_4 - J_5^2 J_4 > 0$.

**Theorem (2):** (i) The system in the absence of spatiotemporal attributes at the inner steady state $P_0(x, y, z, w)$ attains steadiness, then the corresponding uniform steady state of the model 1-4 in the presence of spatiotemporal attributes also attains steadiness. (ii) If the inner steady state $P_0(x, y, z, w)$ of the non-spatial heterogeneity system is unstable, then the respective steady state of the spatiotemporal model 1-4 under initial and boundary settings and attain steadiness by increasing or decreasing the spatiotemporal attributes suitably.

**Proof:** Let us define the function

$V(t) = \int_0^t V(x, y, z, w) dt$, where $V(x, y, z, w)$ is defined in Stability analysis section. Differentiating $V(t)$ w.r.t to $t$ along the solutions of the diffusive model (2.1)-(2.4), we get,

$V'(t) = \left( \int_0^t \sum_{i=1}^4 V_i(x_i(t), y_i(t), z_i(t), w_i(t)) dt \right) = I_R + I_D$ (18)

where $I_R = \int_0^t V(t) dx$;

$I_D = \int_0^t \left( D_1 x u_x + D_2 y u_y + D_3 z u_z + D_4 w u_w \right) du$ (19)

Using the analysis in [14], we get $I_D = -D_1 \int_0^t \left( x^2 u_x \right) du - D_2 \int_0^t \left( y^2 u_y \right) du - D_3 \int_0^t \left( z^2 u_z \right) du - D_4 \int_0^t \left( w^2 u_w \right) du$ (20)

From 7, 8 and 9, it can clearly be observed that if $I_R < 0$ then $V'(t)$ is negative. If $I_R > 0$ then it is clearly showing if there is an increase in the spatiotemporal attributes $D_1, D_2, D_3$ and $D_4$ adequately huge numeral, $V'(t)$ as negative. Henceforth the succeeding portion of the theorem grasps.

**5. NUMERICAL SIMULATIONS**

In this division, we established the analytical findings through numerical simulations using MATLAB.

**Figure-2(a).** Numerical simulation.

**Figure-2(b).**

**Figure-2(b) denotes the variation of populace against time with parameters**
\[ r = 3, \sigma_1 = 1.8, \sigma_2 = 1.6, m_2 = 1.8, m_1 = 1.2, \ E_1 = 0.9, E_2 = 1.2, \]
\[ q_1 = 0.5, q_2 = 0.7, \alpha_1 = 0.5, \alpha_2 = 0.8, \gamma_1 = 0.5, \gamma_2 = 0.2, \ K = 10, s = 3.2; \text{ and initial conditions } x_0 = [10; 10; 5; 5]. \]

Figure-2, denotes the steady fluctuations of the prey, predator populations in both the patches against space and time with \( r = 3, \sigma_1 = 1.8, \sigma_2 = 1.6, m_1 = 1.2, m_2 = 1.8; E_1 = 0.9; E_2 = 1.2, \)
\[ q_1 = 0.5, q_2 = 0.7, \alpha_1 = 0.5, \alpha_2 = 0.8, \gamma_1 = 0.5, \gamma_2 = 0.2, \ K = 2.5, s = 3.2, \]
\[ L = 3; \]
Figures above denotes the steady fluctuations of the prey, predator populations in both the patches against space and time with $r = 5, \sigma_1 = 1.8, \sigma_2 = 2, m_1 = 1.2, m_2 = 1.8, E_1 = 0.9, E_2 = 1.2; q_1 = 0.5; q_2 = 0.7; \alpha_1 = 0.5, \alpha_2 = 0.8; \gamma_1 = 0.5, \gamma_2 = 0.2, L = 3, K = 2.5, s = 3.2$.

6. CONCLUSIONS

In this paper, a mathematical model has been proposed and analyzed to study the stability on the dynamics of a two patchy predator-prey system. The model has been analyzed in a marine environment with effect of harvesting for both prey species. Initially we have discussed about the model and investigated the existence of equilibrium points, local stability by employing Routh-Hurwitz criteria, global analysis by constructing Lyapunov function. Later, we discussed about the dynamics of the diffusion model computer simulations with MATLAB have been executed to study the effects of various parameters on the dynamics of the system. The analytical results and numerical simulation of deterministic model suggest that the deterministic prey predator model is stable in nature. The stability of the system and variations in growth rate for the population species for various parameters shows in figures 2(a), 2(b) and the figures (5.1)-(5.8) represents the variation of populations against time and space.

REFERENCES


