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DISCRETIZED BOUNDARY INTEGRAL EQUATIONS OF UNSATURATED EXPANSIVE SOILS IN TWO DIMENSIONS

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ABSTRACT

The boundary integral equations constitute a remarkably elegant formal solution of the elastoplastic problem of unsaturated expansive soils. Numerical techniques (Boundary element method) must be employed to solve them. The main difficulty of this nonlinear analysis is that the boundary integral equations are augmented by a domain integrals involving initial stresses which necessitate domain discretization and the specification of interior cells in the parts of the domain that are likely to yield (unlike the linear case which involve only the boundary discretization). The main purpose of this paper is to write the discretized form of boundary integral equations of expansive soils using the shape functions.

Keywords: discretization, boundary integral equation, elastoplastic, expansive soils.

INTRODUCTION

To solve the elastoplastic problem of unsaturated expansive soils, we have to deal with the following boundary integral equations that have been found in previous works (J. El Brahmi *et al.*, 2017):

$$\dot{U}_j(\zeta) = \int_S \left[\dot{t}_i(x) G_{ij}(x,\zeta) - F_{ij}(x,\zeta) \dot{U}_i(x) \right] dS + \int_V \mathrm{T}\left(1+2\ln r\right) y_j \nabla^2\left(\dot{s}\right) dV + \int_S \mathrm{T}\left(\frac{2}{r^2} y_j y_m n_m + (1+2\ln r) n_j \right) \dot{s}\, dS$$
$$- \int_S \mathrm{T}\left(1+2\ln r\right) y_j \frac{\partial \dot{s}}{\partial n} dS + \int_V \frac{\partial G_{ij}(x,\zeta_0)}{\partial x_k} \left(\dot{\sigma}^0_{ik} - S_{ik} ds \right) dV \qquad (1)$$

where:

ζ et x : field point coordinate and source point coordinate respectively

$\dot{t}_i$ et $\dot{U}_i$: traction increment and displacement increment respectively on the boundary S of the domain

G_{ij} et F_{ij} : kernel Functions (the Kelvin fundamental solutions generated by unit loads in an infinite elastic solid)

$\nabla^2\left(\dot{s}\right)$, $\dot{s}$ et $\frac{\partial \dot{s}}{\partial n}$: Laplacian of succion, suction increment and gradient of suction increment.

r : distance between the field point and the source point given by : $r^2 = y_i y_i = \left(x_1 - \zeta_1\right)^2 + \left(x_2 - \zeta_2\right)^2$ où $y_i = x_i - \zeta_i$

T : a scalar given by $\Gamma = \left(\lambda + \frac{2}{3}\mu\right) \frac{\kappa_s}{1+e} . \frac{1}{s + p_{atm}}$

where:

λ , μ : Lamé constants

κ_s : compressibility coefficient when succion increments are in the elastic range

e : void ratio

p_{atm} : atmospheric pressure

S and V : boundary surface and domain integral

$\dot{\sigma}^0_{ik} - S_{ik} ds$: initial stresses (plastic part of stresses) given by:

$$\dot{\sigma}^0_{ij} = \frac{\left(\frac{\partial F}{\partial \sigma_{ij}}\right)^T \left(\lambda \delta_{ij} \dot{\varepsilon}_{kk} + 2\mu \dot{\varepsilon}_{ij}\right) \left(\lambda \delta_{ij} \frac{\partial F}{\partial \sigma_{kk}} + 2\mu \frac{\partial F}{\partial \sigma_{ij}}\right)}{H + \left(\frac{\partial F}{\partial \sigma_{ij}}\right)^T \left(\lambda \delta_{ij} \frac{\partial F}{\partial \sigma_{kk}} + 2\mu \frac{\partial F}{\partial \sigma_{ij}}\right)}$$

and $S_{ij} = \left(\frac{\left(\left(\frac{\partial F}{\partial \sigma_{ij}}\right)^T \Gamma \delta_{ij} + \frac{\partial F}{\partial s} \right) \left(\lambda \delta_{ij} \frac{\partial F}{\partial \sigma_{kk}} + 2\mu \frac{\partial F}{\partial \sigma_{ij}}\right)}{H + \left(\frac{\partial F}{\partial \sigma_{ij}}\right)^T \left(\lambda \delta_{ij} \frac{\partial F}{\partial \sigma_{kk}} + 2\mu \frac{\partial F}{\partial \sigma_{ij}}\right)} \right)$

Despite the apparent mathematical complexity of these boundary integral equations, it constitutes a remarkably elegant formal solution of the governing elastoplastic equations of expansive soils. Unfortunately, this integral equations defies solution by analytical means. In practice, numerical techniques (Boundary element method) must be employed to solve it.

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NUMERICAL IMPLEMENTATION IN TWO DIMENSIONS

The integral equations (1) consist of two mechanical integrals:

$$\int_S \left[t_i(x) G_{ij}(x,\zeta) - F_{ij}(x,\zeta) u_i(x) \right] dS \quad (2)$$

$$\int_V \frac{\partial G_{ij}(x,\zeta_0)}{\partial x_k} \left(\sigma_{ik}^0 - S_{ik} ds \right) dV \quad (3)$$

where the first one (equation 2) treats the elastic part of the problem and needs the discretization of the boundary S of the problem, while the second integral (equation 3) deals with the plastic stresses (initial stresses). The equation (3) cannot be conveniently transformed into a boundary integral, which necessitate domain discretization and the specification of interior cells in the parts of the domain that are likely to yield (because the domain integrals are zero elsewhere).

The remaining integrals of equation (1), which are related to the moisture flow (soil suction), are as follows:

$$\int_V T\left(1 + 2\ln r\right) y_j \nabla^2 \left(\right) dV \quad (4)$$

$$\int_S T\left(\frac{2}{r^2} y_j y_m n_m + (1+2\ln r) n_j \right) dS \quad (5)$$

$$\int_S T\left(1 + 2\ln r\right) y_j \frac{\partial}{\partial n} dS \quad (6)$$

The integrals involving succion increments and succion gradient increments (equations 5 and 6) involve only the discretization of the boundary S of the problem. The first integral (equation 4) could be determined analytically.

The main difficulty of this nonlinear analysis is that the boundary integral equations are augmented by those domain integrals involving initial stresses which it is one obvious difference between the linear and nonlinear cases.

Boundary and domain discretization

The boundary "S" of the region of interest "V" is subdivided into a sufficient number (N_e) of elements (which should form a piecewise continuous approximation to the boundary).In each element, the global coordinates x_i are interpolated between the coordinates x_i^α of the nodes of that element through "one dimensional" interpolation functions thus:

$$x_i = \sum_{\alpha=1}^{3} N_\alpha(\xi) . x_i^\alpha$$

For quadratic interpolation, three nodes $(\alpha = 1,2,3)$ must be defined for each element as depicted in Figure-1. and the parameters ξ is the local (intrinsic) coordinate, defined by the curvilinear axis system that is everywhere tangential to the element and normally take values - $-1, 0, 1$ (Figure-1.b)

The interpolation functions $N_\alpha(\xi)$ which are commonly referred to as "shape functions" can be expressed as:

$$N_1(\xi) = \frac{1}{2}\xi\left(\xi - 1\right)$$

$$N_2(\xi) = \frac{1}{2}\xi\left(\xi + 1\right)$$

$$N_3(\xi) = 1 - \xi^2$$

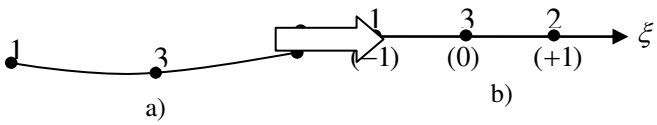

Figure-1. Serendipity line elements (quadratic case).

The discretization requires also division of the yield region into (N_c) iso-parametric cells. In each cell, the geometry and initial stresses are interpolated between the nodes using shape functions, that is,

$$x_i = \sum_{\alpha=1}^{8} M_\alpha(\xi,\eta) . x_i^\alpha$$

$$\sigma_{ij}^p = \sum_{\alpha=1}^{8} M_\alpha(\xi,\eta) . \sigma_{ij}^{p\alpha}$$

where $\sigma_{ij}^{p\alpha}$ is the ij[th] component of the initial stress at node α, and $M_\alpha(\xi,\eta)$ are the two-dimensional shape functions.

For quadratic interpolation, eight nodes $(\alpha = 1,2,..8)$ must be defined for each cell as depicted in Figure-2.a (four nodes at the corners of the intrinsic element and four nodes at the center of each side). The shape functions for the corner nodes $(\alpha = 1,2,3,4)$ are:

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$$M_\alpha(\xi,\eta)=\frac{1}{4}\left(1+\xi_\alpha\xi\right)\left(1+\eta_\alpha\eta\right)\left(-1+\xi_\alpha\xi+\eta_\alpha\eta\right) \qquad M_\alpha(\xi,\eta)=\frac{1}{4}\left(1+\xi_\alpha\xi+\eta_\alpha\eta\right)\left(1-\left(\xi_\alpha\eta\right)^2-\left(\eta_\alpha\xi\right)^2\right)$$

and those for the mid-side nodes ($\alpha=5,6,7,8$) are:

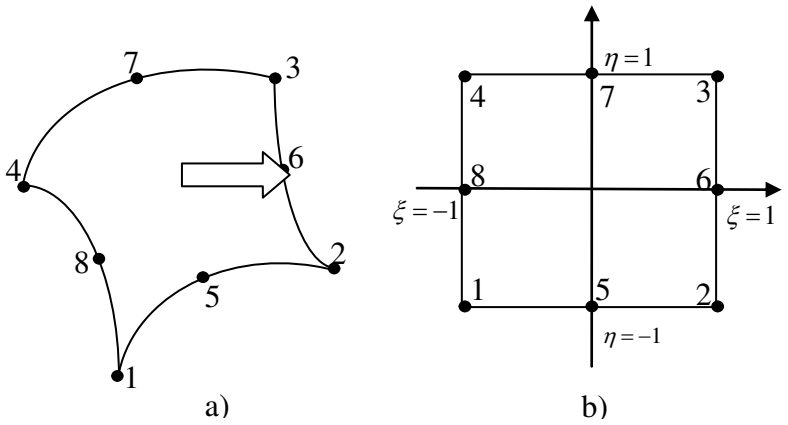

Figure-2. Serendipity cells (quadratic case).

The parameters ξ and η ,are the local (intrinsic) coordinates, defined by the curvilinear axis system that is everywhere tangential to the element and they normally take values in the range ± 1 (Figure-2.b).

Interpolation of tractions and displacements:

The displacements and tractions at intrinsec coordinates within an element are defined in terms of the element nodal values of displacements and tractions using interpolation functions. In "iso-parametric" formulation, we obtain:

$$U_i(\xi)=\sum_{\alpha=1}^{3}N_\alpha\left(\xi\right).\mathrm{U}_i^\alpha$$

$$t_i^{'}(\xi)=\sum_{\alpha=1}^{3}N_\alpha\left(\xi\right).\mathrm{t}_i^{'\alpha}$$

where α denotes the α^{th} node in the 3-noded element.

Discretized boundary integral equations

Once all of boundary conditions and initial stresses are known, we can write the equations (2),(3),(4),(5) and (6) using the shape functions as follows respectively:

$$\int_S\left[t_i(x)G_{ij}(x,\zeta)-F_{ij}(x,\zeta)U_i(x)\right]dS=\sum_{e=1}^{N_e}\left\{\sum_{\alpha=1}^{3}t_i^\alpha(x)\int_{S_e}G_{ij}(x,\zeta)\mathrm{N}_\alpha(\mathrm{x})\mathrm{dS}(\mathrm{x})\right\}-\sum_{e=1}^{N_e}\left\{\sum_{\alpha=1}^{3}U_i^\alpha(x)\int_{S_e}F_{ij}(x,\zeta)\mathrm{N}_\alpha(\mathrm{x})\mathrm{dS}(\mathrm{x})\right\}$$

$$\int_V\frac{\partial G_{ij}(x,\zeta_0)}{\partial x_k}\left(\sigma_{ik}^0-S_{ik}ds\right)dV=\sum_{c=1}^{N_c}\sum_{\alpha=1}^{8}\sigma_{ik}^{p\alpha}\int_{V_c}E_{ijk}\left(x,\zeta\right)M_\alpha(\mathrm{x})\mathrm{dV}(\mathrm{x})$$

$$\int_V\mathrm{T}\left(1+2\ln r\right)y_j\nabla^2\left(\psi\right)dV=\Gamma\sum_{c=1}^{N_c}\sum_{\alpha=1}^{8}(\psi)^\alpha\int_{V_c}\mathrm{b}.\mathrm{y}_j.\mathrm{M}_\alpha(\mathrm{x})\mathrm{dV}(\mathrm{x})$$

$$\int_S\mathrm{T}\left(\frac{2}{r^2}y_jy_mn_m+(1+2\ln r)n_j\right)dS=\Gamma\left(\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}s^\alpha\int_{S_e}(a_j+b.n_j)\mathrm{N}_\alpha(\mathrm{x})\mathrm{dS}(\mathrm{x})\right)$$

$$\int_S\mathrm{T}\left(1+2\ln r\right)y_j\frac{\partial\psi}{\partial n}dS=\Gamma\left(\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}\left(\frac{\partial s}{\partial n}\right)^\alpha\int_{S_e}\mathrm{b}_j.\mathrm{N}_\alpha(\mathrm{x})\,\mathrm{dS}(\mathrm{x})\right)$$

in which V_c and S_e are the domain of the *cth* cell and the length of the *eth* element respectively and

$$E_{ijk}\left(x,\zeta\right)=\frac{\partial G_{ij}(x,\zeta_0)}{\partial x_k}$$

$$\mathrm{b}=\left(1+2\ln r\right)$$

$$\mathrm{b}_j=\mathrm{b}.\mathrm{y}_j=\left(1+2\ln r\right)y_j$$

$$a_j=\frac{2}{r^2}y_jy_mn_m$$

The equation (1) becomes:

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$$C_{ij}\dot{u}_j(\zeta)=\sum_{e=1}^{N_e}\left\{\sum_{\alpha=1}^{3}t_i^{\alpha}(x)\int_{S_e}G_{ij}(x,\zeta)\mathrm{N}_{\alpha}(\mathrm{x})\mathrm{dS}(\mathrm{x})\right\}-\sum_{e=1}^{N_e}\left\{\sum_{\alpha=1}^{3}U_i^{\alpha}(x)\int_{S_e}F_{ij}(x,\zeta)\mathrm{N}_{\alpha}(\mathrm{x})\mathrm{dS}(\mathrm{x})\right\}$$
$$+\Gamma\left(\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}s^{\alpha}\int_{S_e}(a_{j+}b.n_j)\mathrm{N}_{\alpha}(\mathrm{x})\mathrm{dS}(\mathrm{x})-\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}\left(\frac{\partial s}{\partial n}\right)^{\alpha}\int_{S_e}\mathrm{b.y}_j.\mathrm{N}_{\alpha}(\mathrm{x})\mathrm{dS}(\mathrm{x})\right) \quad (7)$$
$$-\Gamma\sum_{c=1}^{N_c}\sum_{\alpha=1}^{8}(\psi)^{\alpha}\int_{V_c}\mathrm{b.y}_j.\mathrm{M}_{\alpha}(\mathrm{x})\mathrm{dV}(\mathrm{x})+\sum_{c=1}^{N_c}\sum_{\alpha=1}^{8}\dot{u}_{ik}^{p\alpha}\int_{V_c}E_{ijk}(x,\zeta)\,\mathrm{M}_{\alpha}(\mathrm{x})dV(x)$$

To produce a closed set of equations, we choose to write equation (7) for each node in turn; that is, we collocate at each of the nodes. It may be observed that the kernel function – shape function products are carried out over each of the elements or cells. The numerical Gauss quadrature method is employed to carry out these integrations when the source point p(ξ) is not one of the cell's nodes.

To apply the Gauss quadrature rules to an arbitrary interval, it is only necessary to map that interval into Gauss quadrature space, denoted by the symbol ξ - η, with due consideration for the scaling factor (Jacobian) that this introduces. The mapping from the real interval to Gauss quadrature space yields:

$$C_{ij}\dot{u}_j(\zeta)=\sum_{e=1}^{N_e}\left\{\sum_{\alpha=1}^{3}t_i^{\alpha}\int_{-1}^{1}G_{ij}(\xi,\zeta)N_{\alpha}(\xi)J_e(\xi)d\xi\right\}-\sum_{e=1}^{N_e}\left\{\sum_{\alpha=1}^{3}U_i^{\alpha}\int_{-1}^{1}F_{ij}(\xi,\zeta)N_{\alpha}(\xi)J_e(\xi)d\xi\right\}$$
$$+\Gamma\left(\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}s^{\alpha}\int_{-1}^{1}(a_j+b.n_j)N_{\alpha}(\xi)J_e(\xi)d\xi-\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}\left(\frac{\partial s}{\partial n}\right)^{\alpha}\int_{-1}^{1}\mathrm{b.y}_j\,N_{\alpha}(\xi)J_e(\xi)d\xi\right) \quad (8)$$
$$-\Gamma\sum_{c=1}^{N_c}\sum_{\alpha=1}^{8}(\psi)^{\alpha}\int_{-1}^{1}\int_{-1}^{1}\mathrm{b.y}_j\,N_{\alpha}(\xi,\eta)J_c(\xi,\eta)d\xi d\eta+\sum_{c=1}^{N_c}\sum_{\alpha=1}^{8}\dot{u}_{ik}^{p\alpha}\int_{-1}^{1}\int_{-1}^{1}E_{ijk}(\xi,\zeta)N_{\alpha}(\xi,\eta)J_c(\xi,\eta)d\xi d\eta$$

where $J_e(\xi)$ and $J_c(\xi,\eta)$ are the Jacobien of transformation:

$$J_e(\xi)=\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial y}{\partial \xi}\right)^2}$$
$$J_c(\xi,\eta)=\frac{\partial x}{\partial \xi}.\frac{\partial y}{\partial \eta}-\frac{\partial y}{\partial \xi}.\frac{\partial x}{\partial \eta}$$

or in more reduced form:

$$C_{ij}\dot{u}_j(\zeta)=\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}t_i^{\alpha}G_{ij}^{e}-\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}U_i^{\alpha}F_{ij}^{e}+\Gamma\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}s^{\alpha}A^{e}-\Gamma\sum_{e=1}^{N_e}\sum_{\alpha=1}^{3}\left(\frac{\partial s}{\partial n}\right)^{\alpha}B^{e} \quad (9)$$
$$-\Gamma\sum_{c=1}^{N_c}\sum_{\alpha=1}^{8}(\psi)^{\alpha}B^{c}+\sum_{c=1}^{N_c}\sum_{\alpha=1}^{8}\dot{u}_{ik}^{p\alpha}E_{ijk}^{c\alpha}$$

where:

$$G_{ij}^{e}=\int_{S_e}G_{ij}(x,\zeta)\,\mathrm{N}_{\alpha}(\mathrm{x})\,\mathrm{dS}(\mathrm{x})=\int_{-1}^{1}G_{ij}(x,\zeta)N_{\alpha}(\xi)J_e(\xi)d\xi$$
$$F_{ij}^{e}=\int_{S_e}F_{ij}(x,\zeta)\,\mathrm{N}_{\alpha}(\mathrm{x})\,\mathrm{dS}(\mathrm{x})=\int_{-1}^{1}F_{ij}(x,\xi)N_{\alpha}(\xi)J_e(\xi)d\xi$$

$$A^{e}=\int_{S_e}(a_j+b.n_j)\,\mathrm{N}_{\alpha}(\mathrm{x})\,\mathrm{dS}(\mathrm{x})=\int_{-1}^{1}(a_j+b.n_j)N_{\alpha}(\xi)J_e(\xi)d\xi$$

$$B^{e}=\int_{S_e}\mathrm{b.y}_j.\mathrm{N}_{\alpha}(\mathrm{x})\,\mathrm{dS}(\mathrm{x})=\int_{-1}^{1}\mathrm{b.y}_j\,N_{\alpha}(\xi)J_e(\xi)d\xi$$

$$B^{c}=\int_{V_c}\mathrm{b.y}_j.\mathrm{N}_{\alpha}(\mathrm{x})\,\mathrm{dV}(\mathrm{x})=\int_{-1}^{1}\int_{-1}^{1}\mathrm{b.y}_j\,N_{\alpha}(\xi,\eta)J_c(\xi,\eta)d\xi d\eta$$

$$E_{ijk}^{c\alpha}=\int_{V_c}E_{ijk}(x,\zeta)\,\mathrm{N}_{\alpha}(\mathrm{x})\,\mathrm{dV}(\mathrm{x})=\int_{-1}^{1}\int_{-1}^{1}E_{ijk}(x,\xi)N_{\alpha}(\xi,\eta)J_c(\xi,\eta)d\xi d\eta$$

It is to highlight that when the field point $\mathrm{P}(\zeta)$ is located in the same element as the source point $Q(\mathrm{x})$, the G_{ij} and F_{ij} kernels become singular by the fact that they contain terms of order r^{-1} and r^{-2}, respectively. In this case the direct application of Gaussian quadrature is inadequate and special techniques must be employed to resolve the singularities.

MOISTURE FLOW MODEL

To treat equation (9), we have to determine firstly all of boundary suctions and gradient suctions. Indeed, using total suction as the state variable, the governing diffusion equation for steady state can be written as (Zoukaghe, 1985),

$$\nabla^2 s=-\psi \quad (10)$$

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where $\psi = \frac{Q}{K}$

with Q is the strength of the source (if there is any) expressed as rate of suction generation per unit volume, and K is the saturated permeability (if isotropic permeability is considered).

Equation (10), poisson's equation, is assumed to be sufficiently differentiable for the Laplacian ∇^2 to exist. The solution of that equation defines the distribution of suction throughout the soil mass as a function of location. This solution can be uniquely determined by proper prescription of the boundary conditions (soil suction s and/or gradient of suction $\frac{\partial s}{\partial n}$). Numerical solutions to these equations are quite popular in both research and industrial environments. In these areas, Finite Difference and finite Element methods are used extensively. However, the boundary Integral Equations Method will be used again to solve them.

The boundary integral formulation to the diffusion equation (10) subjected to specified boundary conditions is:

$$\alpha s(\zeta) = \int_S G \frac{\partial s}{\partial n} dS - \int_S F s dS + \int_V G \psi dV \qquad (11)$$

This equation (11) defines soil suction s at any boundary or interior point ζ, where The function G (which represents the effect at a field point X of a unit source applied at a point ζ)is given by $G = -\frac{1}{2\pi}\ln(r)$ and its associated gradient being $F = -\frac{1}{2\pi r^2} y_i n_i$, where r, y_i and n_i are as defined previously.

The coefficient α is related to the geometry of the problem. When ξ is an internal point, then (α=1); when the point ξ lies on a smooth boundary, then (α=1/2).

To compute soil suction s at interior points of the domain, it is necessary to resolve the boundary problem first. Obviously, at a point on the boundary S, either s or $\frac{\partial s}{\partial n}$ is known a priori in a well posed problem (boundary conditions), and by writing the equation (11) for each node of the boundary S in turn; we collocate at each of the nodes producing a closed set of equations.

CONCLUSIONS

In this paper, we have sought to demonstrate the principal techniques that must be employed to translate the formal boundary integral equation solutions of unsaturated expansive soils into a practical numerical method. A computer program that embodies these techniques will be implemented in future papers.

REFERENCES

El Brahmi J., Lamdouar N. and Zoukaghe M. 2017. Derivation of the Incremental Stress-Strain Relations for Expansive Soils and Implementation into the Boundary Element Method. Numerical.

Analysis of Nonlinear Coupled Problems. Proceedings of the 1st GeoMEast International Congress and Exhibition, Egypt 2017 on Sustainable Civil Infrastructures. DOI 10.1007/978-3-319-61905-7_23.

Xiao-Wei Gao and Trevor G. Davies. 2002. Boundary element programming in mechanics. Cambridge university press.

Zoukaghe M. 1985. Moisture diffusion and generated stresses in expansive soils by the boundary element method. Doctoral thesis, Faculty of the Graduate School of the University of Missouri - Rolla.