EVALUATION OF PERFORMANCE MEASURES OF PRIORITY QUEUES WITH FUZZY PARAMETERS USING α - CUT APPROACH

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ABSTRACT

In this paper, we extend the queuing model of priority classes to the fuzzy environment where the uncertainty is resolved by fuzzy set theory. A parametric programming problem is constructed where the Trapezoidal fuzzy numbers are introduced for the arrival rates and service time of the fuzzy queue. By the α-cuts, we obtain crisp values from the fuzzy queues with the help of Zadeh's principle. To establish the performance measure of the model, a numerical example is given.

Keywords: aueuing model, fuzzy sets, membership functions, priority queues, mathematical programming and trapezoidal fuzzy number.
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1. INTRODUCTION

In our day-to-day life there is always a need for some service, for which we approach places where the service is available. But it is not going to be only us who need that service, there might be many who need the some service. The service will be provided for the person who came first at the service station and others who came after that need to wait in the queue. Queueing process is a mechanism where customers arrive at a service counter in a queue. The service is preceded one by one where customers waiting in a prior to them will be attended first. After receiving the service customers will depart from the service centre. Thus a queueing system can be described as a process consisting of customer's arrival, customers waiting in queue, attending the service and leaving the service centre. It is a type of facility where customers are provided with service when there is a formation of a queue. The service is of different types depending on the arrival or priority of the customers.

1.1 Priority discipline

In Priority discipline, customer service is done according to the priority where a customer with higher priority is preferred over a customer with lower priority.

1.2 Fuzzy set theory

In practical situation, input data like arrival and service rates are imprecise in this paper. So in order to resolve this uncertainty fuzzy set theory is used. In mathematics, the sets where the membership are in degrees. The crisp sets are usually called as classical bivalent sets. For imprecise inputs fuzzy set theory is used to resolve them.

1.3 Fuzzy priority queues

In this paper, membership function of the fuzzy priority queues is constructed by applying fuzzy set theory. Trapezoidal fuzzy numbers are employed for the arrival and service rates. The characteristics of the system are drawn from α-cut and fuzzy arithmetic operations.

1.4 Fuzzy set

A fuzzy set is a pair (X, A) where X is a set and A: X→ [0, 1]. A fuzzy set is a set where the members are allowed to have partial membership and hence the degree of membership varies from 0 to 1.

1.5 α- CUT:

α- cut of a fuzzy set A is the crisp set where the members have their membership values greater than or at least equal to α.

1.6 Fuzzy number

Fuzzy number is expressed as a fuzzy set defining a fuzzy interval in the real number $\mathbb{R}$. Fuzzy number should be normalized and convex.

1.7 Trapezoidal fuzzy number

Trapezoidal fuzzy number can be defined as A = (a_1, a_2, a_3, a_4) the membership function of this fuzzy number will be interpreted as follows

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \le x \le a_4 \\ 0, & x > a_4 \end{cases}$$

Figure-3.1. Trapezoidal fuzzy number.

A= (a_1, a_2, a_3, a_4)

1.8 α- cut of trapezoidal fuzzy number

The α- cut interval for this shape is given as

$A_\alpha = [(a_2 - a_1)\alpha + a_1, \quad -(a_4 - a_3)\alpha + a_4]$

2. LITERATURE REVIEW

In this model, it is obvious that the arrival rate, the service rate, the input data are uncertain. So in order to solve this uncertainty, we approach fuzzy where it is employed to our model. Many researchers have illustrated this type of model. [Kao, Chenand Li] used inter-arrival time and service time to compose the membership functions and formulate parametric program. [Lee and Li] presented a conventional method in the fuzzy situation based on Principle of Zadeh in the queueing system. [Negi and Lee] used trapezoidal fuzzy number to characterize variables and showed analytical results by means of those numbers. [Zadeh] defined the concept of possibility distribution in the fuzzy sets as constraint in his paper "Fuzzy sets as a basis for a theory of possibility. [Zimmermann] considered membership function of the fuzzy queues where the uncertainties are Trapezoidal fuzzy number. Then with different α-cut approach we derive the system characteristics. [Ritha and Robert] proposed a priority model where with the help of parametric program and trapezoidal fuzzy number to find the analytical results.

3. DEVELOPMENT OF PARAMETRIC PROGRAMMING PROBLEM FOR FM/FM/I FUZZY QUEUE WITH PRIORITY MODEL

In order to demonstrate our model we consider a fuzzy queueing model FM/FM/I with priority discipline.

3.1 Representation

In our fuzzy queueing model we represent our uncertainties using fuzzy set theory.

3.2 Fuzzy sets

The uncertainties of the model are the inter-arrival times of the higher and lower priority and service times. They are denoted as $\widetilde{AT}_k$, k=1,2 and $\widetilde{ST}$ respectively.

They are represented by the following fuzzy sets

$$\widetilde{AT}_k=\{(a, \mu_{\widetilde{AT_K}}(a))/a\epsilon U\}, \quad k = 1,2 \tag{1}$$

$$\widetilde{ST} = \{(s, \mu_{\widetilde{ST}}(s))/s\epsilon V\} \tag{2}$$

U and V are crisp sets of $\widetilde{AT}_k$, k=1,2 and $\widetilde{ST}$ respectively.

$\mu_{\widetilde{AT_k}}(a)$ and $\mu_{\widetilde{ST}}(s)$ are respective membership functions.

3.3 α- cut

The α- cuts are given for the proposed model

$$AT_k(\alpha)= \{a\epsilon U/\mu_{\widetilde{AT_k}}(a)\geq\alpha\}, \text{ k=1,2} \tag{3}$$

$$\text{ST}(\alpha)= \{s\epsilon V/\mu_{\widetilde{ST}}(s) \geq\alpha\}, \tag{4}$$

Where 0<α≤1. Both $AT_k(\alpha)$ and ST(α) are crisp sets.

3.4 Confidence intervals

To estimate the value of parameter, $\widetilde{AT}_k$ and $\widetilde{ST}$ is represented by different levels confidence intervals [0,1] using α-cuts. The α-cuts are given by $\{AT_k(\alpha)/$ 0<α≤1},k=1,2 and {ST(α)/ 0<α≤1}. Using these α-cuts, fuzzy queue is reduced to family of crisp queues.

The confidence interval to be estimated is given for $\widetilde{AT}_k$ and $\widetilde{ST}$ be $\left[l_{AT_{k(\alpha)}}, u_{AT_{k(\alpha)}}\right]$, k = 1,2 and $\left[l_{ST(\alpha)}, u_{ST(\alpha)}\right]$ respectively.

3.5 Use of Zadeh's extension principle

With the help of extension principle of Zadeh, the performance measure's membership function is given by pm ($\widetilde{AT}_k, \widetilde{ST}$), k=1, 2 is defined as

$$\mu_{\text{pm}(\widetilde{AT_k},\widetilde{ST})}(z) = sup_{a\epsilon X, s\epsilon Y} \min\{\mu_{\widetilde{AT_k}}(a),\mu_{\widetilde{ST}}(s)/z=\text{pm}(a,s)\}, \text{ k=1,2} \tag{5}$$

From equation (5)

$\mu_{\text{pm}(\widetilde{AT_k},\widetilde{ST})}(z)$ =α, k=1,2 is true only when either $\mu_{\widetilde{AT_k}}(a)$=α, $\mu_{\widetilde{ST}}(s)$ ≥α (or) $\mu_{\widetilde{AT_k}}(a)$ ≥α, $\mu_{\widetilde{ST}}(s)$ =α is true.

3.6 Parametric programming problem

After considering the fuzzy sets of inter-arrival times and service time and their membership functions, different levels of confidence intervals are given to the inter-arrival times and service time. Using Zadeh's extension principle, performance measure's membership function is defined. From these considerations, we obtain the following parametric programming problem

$$l_{pm(\alpha)} = \text{MIN}\, pm(a, s) \tag{6}$$

Such that $l_{AT_{k(\alpha)}} \leq a \leq u_{AT_{k(\alpha)}}$, k=1,2

$l_{ST(\alpha)} \leq s \leq u_{ST(\alpha)}$

And

$$u_{pm(\alpha)} = \text{MAX}\,\text{pm}(a, s) \tag{7}$$

Such that $l_{AT_{k(\alpha)}} \leq a \leq u_{AT_{k(\alpha)}}$, k=1,2

$l_{ST(\alpha)} \leq s \leq u_{ST(\alpha)}$

3.7 Shape function

The membership function $\mu_{\text{pm}(\widetilde{At_k},\widetilde{ST})}(x)$, k=1,2 is constructed as

$$\mu_{\text{pm}(\widetilde{AT_k},\widetilde{ST})}(z) \begin{Bmatrix} LS(x), & for\ x_1 \leq x \leq x_2 \\ RS(x), & for\ x_3 \leq x \leq x_4 \\ 0\,, & otherwise \end{Bmatrix} \tag{8}$$

Where $x_1 \leq x_2 \leq x_3 \leq x_4$ and LS(x_1) = RS(x_4)=0 for the trapezoidal fuzzy number.

The left shape function L(x) = $(l_{pm(\alpha)})^{-1}$ and the right shape R(x) =$(u_{pm(\alpha)})^{-1}$ can be achievedif both $l_{pm(\alpha)}$ and $u_{pm(\alpha)}$are invertible with respect to α.

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3.8 FM/FM/I reduced to M/M/I

In this model FM/FM/I demoted to M/M/I by using the concept of α- cut of fuzzy set theory for which

$Lh_{q_1} = \frac{\rho\frac{\lambda_1}{\mu}}{\left(1-\frac{\lambda_1}{\mu}\right)}$, $Ll_{q_2} = \frac{\left(\rho\frac{\lambda_2}{\mu}\right)}{\left(1-\rho\right)\left(1-\frac{\lambda_1}{\mu}\right)}$

$Wh_{q_1} = \frac{\lambda}{\mu(\mu-\lambda_1)}$, $Wl_{q_2} = \frac{\lambda}{(\mu-\lambda)(\mu-\lambda_1)}$

Further $\lambda=\lambda_1+\lambda_2$ and $\rho = \frac{\lambda}{\mu}$

The membership functions $\mu_{\mathrm{pm}(\widetilde{AT_k},\widetilde{ST})}(x)$ is not derived if the functions $l_{pm(\alpha)}$ and $u_{pm(\alpha)}$ are not invertible with respect to α.

The queueing model with priority can be obtained for the membership functions $\mu_{\mathrm{pm}(\widetilde{AT_k},\widetilde{ST})}(x)$ by applying this procedure.

4. NUMERICAL ILLUSTRATION

To find the probable number of customers and waiting time in the queue with priority classes,

The arrival rates of higher priority - $\widetilde{AT}_1$= [2, 3, 4, 5]

The arrival rates of lower priority -$\widetilde{AT}_2$= [5, 6, 7, 8]

The service rates - $\widetilde{ST}$= [16, 17, 18, 19]

The α-cut of the membership functions is given by

$\mu_{\overline{AT_1}}(\alpha)$ - [2+α, 5-α]
$\mu_{\overline{AT_2}}(\alpha)$ - [6+α, 9-α]
$\mu_{\overline{ST}}(\alpha)$ - [16+α, 19-α]

The parametric programming problem are formulated to derive the membership function $\overline{Lh_{q_1}}$,$\overline{Ll_{q_2}}$, $\overline{Wh_{q_1}}$ and $\overline{Wl_{q_2}}$.

The performance functions of the following are derived from the respective parametric program

$\overline{Lh_{q_1}}$ - average queue length of higher priority
$\overline{Ll_{q_2}}$ - average queue length of lower priority
$\overline{Wh_{q_1}}$- average waiting time of units of higher priority
$\overline{Wl_{q_2}}$- average waiting time of units of lower priority

4.1 Performance functions

The parametric programming problems are from equations (6) and (7) to derive the above membership functions. These functions differ only in their objective functions.

4.2 Objective functions

The objective functions are given below:

$$l_{Lh_{q_1}}(\alpha) = MIN\left\{\frac{\left(\frac{R_1+R_2}{T}\right)\left(\frac{R_1}{T}\right)}{\left(1-\frac{R_1}{T}\right)}\right\} \tag{9}$$

Such that
2+$\alpha \leq R_1 \leq$ 5-α
6+$\alpha \leq R_2 \leq$ 9-α
16+$\alpha \leq T \leq$ 19-α

and

$$u_{Lh_{q_1}}(\alpha) = MAX\left\{\frac{\left(\frac{R_1+R_2}{T}\right)\left(\frac{R_1}{T}\right)}{\left(1-\frac{R_1}{T}\right)}\right\} \tag{10}$$

Such that
2+$\alpha \leq R_1 \leq$ 5-α
6+$\alpha \leq R_2 \leq$ 9-α
16+$\alpha \leq T \leq$ 19-α
Where 0<α≤1

The following objective functions also have the same constraints of the equations (9) and (10)

$$l_{Ll_{q_2}}(\alpha)= \mathrm{MIN}\left\{\frac{\frac{R_2(R_1+R_2)}{T^2}}{\left(1-\frac{R_1+R_2}{T}\right)\left(1-\frac{R_1}{T}\right)}\right\} \tag{11}$$

and $$u_{Ll_{q_2}}(\alpha)= \mathrm{AX}\left\{\frac{\frac{R_2(R_1+R_2)}{T^2}}{\left(1-\frac{R_1+R_2}{T}\right)\left(1-\frac{R_1}{T}\right)}\right\} \tag{12}$$

$$l_{Wh_{q_1}}(\alpha) = \mathrm{MIN}\left\{\frac{R_1+R_2}{T(T-R_1)}\right\} \tag{13}$$

$$u_{Wh_{q_1}}(\alpha) = \mathrm{MAX}\left\{\frac{R_1+R_2}{T(T-R_1)}\right\} \tag{14}$$

$$l_{Wl_{q_2}}(\alpha)= \mathrm{MIN}\left\{\frac{R_1+R_2}{[T-(R_1+R_2)](T-R_1)}\right\} \tag{15}$$

$$u_{Wl_{q_2}}(\alpha)= \mathrm{AX}\left\{\frac{R_1+R_2}{[T-(R_1+R_2)](T-R_1)}\right\} \tag{16}$$

4.3 Optimal solutions

For the higher priority, the optimal solution can be found when R_1 and R_2 approach their lower bounds and T approaches upper bound.

For the lower priority, , the optimal solution can be found when R_1 and R_2 approach their upper bounds and T approaches lower bound.

The optimal solution for (9) is

$$l_{Lh_{q_1}}(\alpha) = \frac{2\alpha^2+12\alpha+16}{2\alpha^2-55\alpha+323} \tag{17}$$

For (10)

$$u_{Lhq_1}(\alpha) = \frac{2\alpha^2-24\alpha+70}{2\alpha^2+43\alpha+176} \tag{18}$$

For (11)

$$l_{Ll_{q_2}}(\alpha)=\frac{2\alpha^2+20\alpha+48}{2\alpha^2-63\alpha+391} \tag{19}$$

For (12)

$$u_{Ll_{q_2}}(\alpha)=\frac{2\alpha^2-32\alpha+126}{2\alpha^2+52\alpha+220} \tag{20}$$

For (13)

$$l_{Wh_{q_1}}(\alpha)=\frac{8+2\alpha}{2\alpha^2-55\alpha+323} \tag{21}$$

For (14)

$$u_{Wh_{q_1}}(\alpha)=\frac{14-2\alpha}{2\alpha^2+43\alpha+176} \tag{22}$$

For (15)

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$$l_{Wl_{q_2}}(\alpha)=\frac{8+2\alpha}{6\alpha^2-73\alpha+187} \quad (23)$$

For (16)

$$u_{Wl_{q_2}}(\alpha)=\frac{14-2\alpha}{6\alpha^2+37\alpha+22} \quad (24)$$

4.4 Estimation of membership functions

The membership function

$$\mu_{\overline{Lh}_{q_1}}(x) = \begin{cases} L(x), & \left[l_{Lh_{q_1}}(\alpha)\right]_{\alpha=0} \le x \le \left[l_{Lh_{q_1}}(\alpha)\right]_{\alpha=1} \\ R(x), & \left[u_{Lh_{q_1}}(\alpha)\right]_{\alpha=1} \le x \le \left[u_{Lh_{q_1}}(\alpha)\right]_{\alpha=0} \\ 0, & otherwise \end{cases}$$

which is estimated as follows

$$\mu_{\overline{Lh}_{q_1}}(x)= \begin{cases} \frac{(55x-12)-(441x^2+4032x+16)^{\frac{1}{2}}}{4(x-1)} & 0.05 \le x \le 0.11 \\ \frac{-(43x-24)+(441x^2+4032x+16)^{\frac{1}{2}}}{4(x-1)} & 0.21 \le x \le 0.39 \\ 0 & otherwise \end{cases}$$

Similarly the membership functions of $\mu_{\overline{Ll}_{q_2}}, \mu_{\overline{Wh}_{q_1}} and \mu_{\overline{Wl}_{q_2}}$ are derived and listed below

$$\mu_{\overline{Ll}_{q_2}}(x)= \begin{cases} \frac{(63x+20)-(841x^2+6032x-16)^{\frac{1}{2}}}{4(z-1)} & 0.12 \le x \le 0.21 \\ \frac{-(51x+32)+(841x^2+6032x-16)^{\frac{1}{2}}}{4(x-1)} & 0.35 \le x \le 0.57 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{\overline{Wh}_{q_1}}(x)= \begin{cases} \frac{(55x+2)-(441x^2+284x+4)^{\frac{1}{2}}}{4x} & 0.02 \le x \le 0.03 \\ \frac{-(43x+2)+(441x^2+284x+4)^{\frac{1}{2}}}{4x} & 0.05 \le x \le 0.07 \\ 0 & otherwise \end{cases}$$

$$\mu_{\overline{Wl}_{q_2}}(x)= \begin{cases} \frac{(73x+2)-(841x^2+484x+4)^{\frac{1}{2}}}{12x} & 0.04 \le x \le 0.08 \\ \frac{-(37x+2)+(841x^2+484x+4)^{\frac{1}{2}}}{12x} & 0.18 \le x \le 0.63 \\ 0 & otherwise \end{cases}$$

5. CONCLUSIONS

In this paper, M/M/I queueing model with priority is considered with fuzzy set theory. By applying fuzzy set theory where the arrival rates and service times are represented by trapezoidal fuzzy number. Also we construct the parametric program by the extension principle of Zadeh. The probable number of customers and waiting time is obtained from the performance measures of the parametric program. To validate our proposal an illustration is given. For future study, the proposed model can be extended with various fuzzy numbers and also it can extend with high priority, Further this model can be extending with more service channels and Even it can be worked with Erlangian service time with K phases

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RESULTS AND DISCUSSIONS

α- Cuts of uncertainties

The following table gives the α- cut values for the uncertainties

α	l_{x_α}	u_{x_α}	l_{y_α}	u_{y_α}	$l_{Lh_{q_1}}(\alpha)$	$u_{Lh_{q_1}}(\alpha)$	$l_{Ll_{q_2}}(\alpha)$	$u_{Ll_{q_2}}(\alpha)$	$l_{Wh_{q_1}}(\alpha)$	$u_{Wh_{q_1}}(\alpha)$	$l_{Wl_{q_2}}(\alpha)$	$u_{Wl_{q_2}}(\alpha)$
0	2	5	16	19	0.0495	0.3977	0.1228	0.5727	0.0248	0.0795	0.0428	0.6364
0.1	2.1	4.9	16.1	18.9	0.0542	0.3750	0.1300	0.5456	0.0258	0.0765	0.0456	0.5357
0.2	2.2	4.8	16.2	18.8	0.0592	0.3535	0.1376	0.5197	0.0269	0.0736	0.0487	0.4588
0.3	2.3	4.7	16.3	18.7	0.0645	0.3331	0.1455	0.4951	0.0280	0.0709	0.0519	0.3983
0.4	2.4	4.6	16.4	18.6	0.0701	0.3138	0.1538	0.4716	0.0292	0.0682	0.0554	0.3496
0.5	2.5	4.5	16.5	18.5	0.0760	0.2955	0.1625	0.4492	0.0304	0.0657	0.0592	0.3095
0.6	2.6	4.4	16.6	18.4	0.0823	0.2781	0.1716	0.4278	0.0316	0.0632	0.0633	0.2761
0.7	2.7	4.3	16.7	18.3	0.0889	0.2616	0.1810	0.4074	0.0329	0.0608	0.0677	0.2478
0.8	2.8	4.2	16.8	18.2	0.0959	0.2460	0.1909	0.3880	0.0342	0.0586	0.0725	0.2237
0.9	2.9	4.1	16.9	18.1	0.1033	0.2312	0.2013	0.3694	0.0356	0.0564	0.0777	0.2028
1	3	4	17	18	0.1111	0.2172	0.2121	0.3516	0.0370	0.0543	0.0833	0.1846

Graphs of performance measures

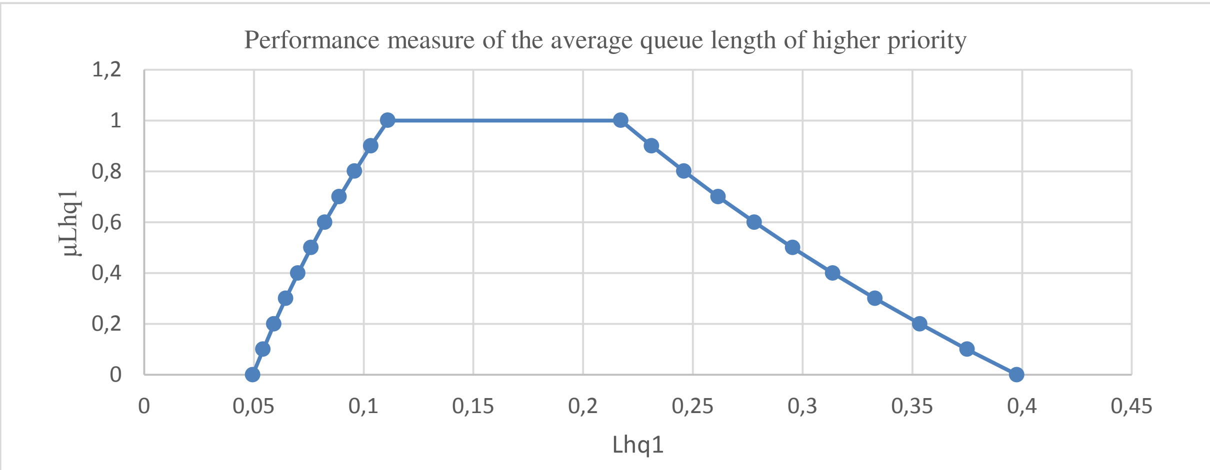

Figure-1.1. Performance measure of the average queue length of higher priority.

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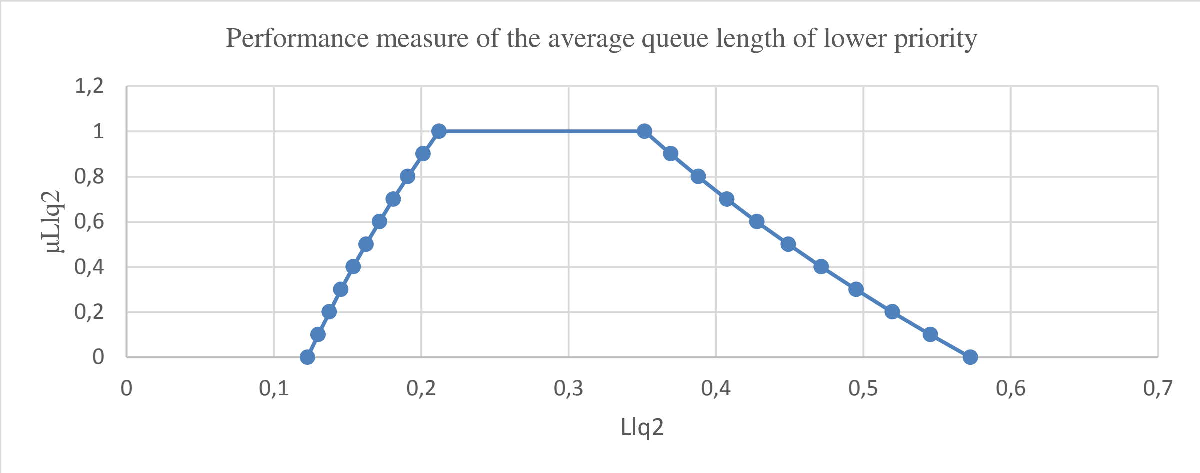

Figure-1.2. Performance measure of the average queue length of lower priority.

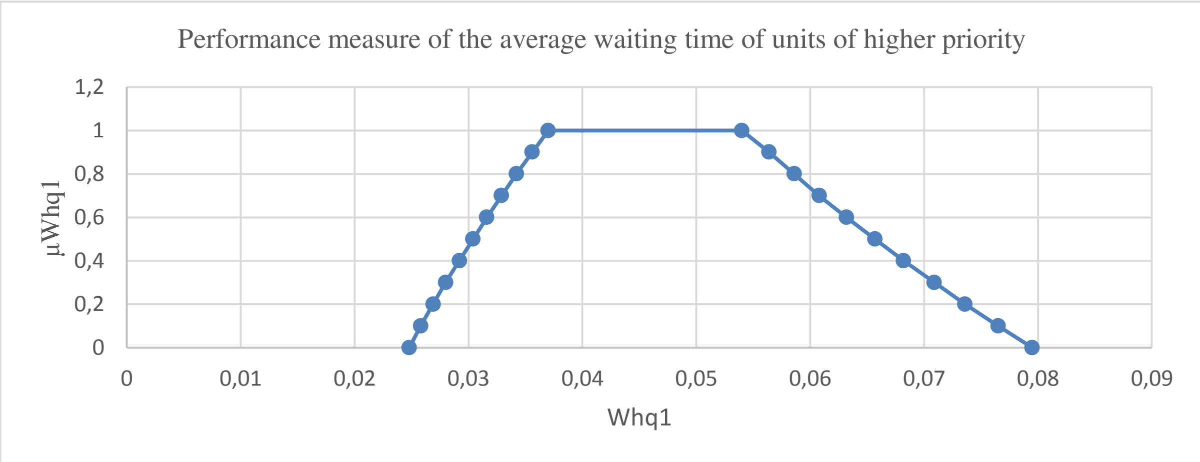

Figure-1.3. Performance measure of the average waiting time of units of higher priority.

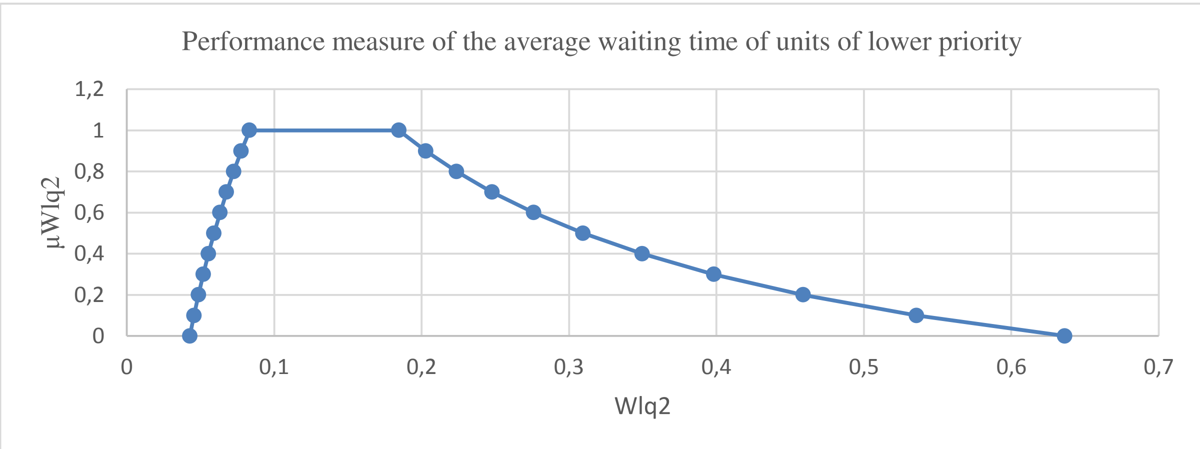

Figure-1.4. Performance measure of the average waiting time of units of lower priority.