A STATE OF THE ART REVIEW ON BEAMS ON NON-UNIFORM ELASTIC FOUNDATIONS

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ABSTRACT

This research is a review on previous work on linear elastic behavior of beams resting on uniform and non-uniform Winkler foundation. Several methods were used previously by other researchers starting from analytical and numerical such as series, finite difference and finite element formulations which are developed to solve various problems. The finite element method is formulated previously by using beam, plate and solid 3D elements to model beams and spring elements to model Winkler foundation in the past. Also, the finite difference coded in matlab and Fortran was used to formulate the problems to deal with various variables using Gauss Jordan and Gauss elimination to solve them and to compare with the results. Also, the Fourier series coded in various computer program to obtain the results and used to check and compare with the results obtained from other methods. In both methods, Fourier series and finite-difference, the formulations are based on the basic differential equations. The results obtained from previous studies using different methods are given and discussed together to compare and check the accuracy of the solutions. Good agreement is found in these methods, which indicate the efficiency of these methods which was used by previous studies.

Keyword: beams, finite difference, finite elements, non-uniform Winkler foundation.

INTRODUCTION

The analysis of linear elastic material behavior of thin beams resting on a linear springs and subjected to distributed and concentrated loads has been studied by several different methods. The beam may have different support end conditions such as fixed, simply supported, free and a load bearing media. Elastic support provided for beams is referred to foundation. In the Winkler model, the ratio of pressure (p) to the displacement (w) is known as modulus of sub-grade reaction (Kz) which is a concept relationship between deflection and soil pressure. The modulus of subgrade reaction is obtained using plate loading test.
The sub-grade (Kz) can be determined using:

$$K_z = \frac{p}{w} \qquad (1)$$

where,
K_z is the modulus of subgrade reaction,
P is the applied pressure
w is the deflection.

Winkler (1867) presented very simplified model, which is now called Winkler foundation. Winkler assumed the soil to have linear elastic behavior, such that at any point the contact pressure to be proportionate to the settlement of the soil. Denoting the pressure at any point by p, and the beam deflection at the same point by **w**. This assumption, often called Winkler's hypothesis .Mathematically it can be written:

$$p(x,y)=K_z\, w(x,y) \qquad (2)$$

For about eighty years since of this hypothesis in the bending of beams on elastic subgrades, most of the investigators in this field worked on solutions of the basic differential equation of the problem. Little attention was given to the question of reliability of the basic hypothesis. However, the investigations performed in the past after Winkler's hypothesis have shown that the distribution of vertical pressure at the contact surface between beams or slabs and elastic subgrades may be quite different from that obtained by the conventional analysis obtained using Winkler's hypothesis. The hypothesis appeared not to be justified at least for beams and slabs on subgrades such as concrete, rock, or soils. Consequently, the coefficient of subgrade reaction (K_z) was considered as an preliminary concept. It appeared that an analysis based on this concept gives approximate estimate.

Biot (1937) solved the problem of infinite beams on a uniform elastic foundation under a concerted load. The elementary Euler- Bernoulli beam theory is used assuming the beam is supported on group of springs. The springs are distributed continuously under the beam. The spring constant is defined through using the resultant of modulus of the foundation K_z.. Generally, the foundation is an elastic continuum characterized by two elastic constants, (modulus of elasticity E and a Poisson ratio ν). The problem of the bending of beams resting on elastic foundation have been solved already by various authors. Biot derived the expressions for shear and bending moment at any point (**x**)of the beam. The author attempts to give a more exact solution of one aspect of this problem, i.e., the case of an infinite beam under a concentrated load. A notable difference exists between the results obtained from the assumptions of 2-dimensional foundation and 3-dimensional foundation. For the 2-dimensional, the deflection and bending-moment curves are shown in Figure-1 and Figure-2. The value of K_z used for 2- and 3-dimensionalcases gives suitable results. The obtained results depend on the stiffness, elasticity of the beam and foundation, It is shown that Winkler's

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hypothesis is practically satisfied for infinite beams (λL>50).

$$\lambda = \sqrt[4]{\frac{K_z}{EI}} \tag{3}$$

where:

L = length of the beam,

EI = flexural rigidity of the beam.

K_z = modules of subgrade reaction.

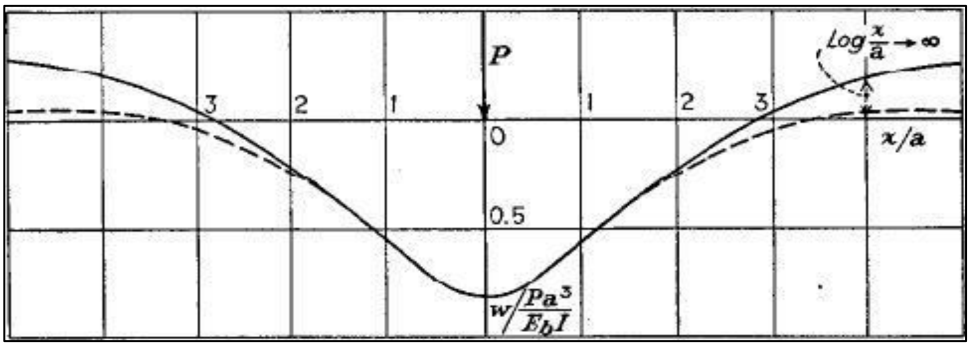

Figure-1. Deflection curves.

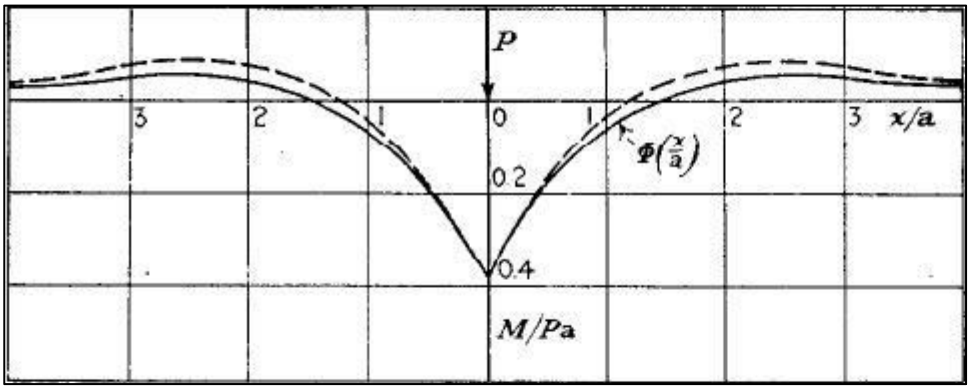

Figure-2. Bending - moment curves

Hetenyi (1946) presented a textbook for the theory and application of elastically supported beams in the field of civil engineering. Hetenyi presented formulas for simple cases of beam of finite length and infinite length. The formulas were derived by the method of superposition. The main purpose of this textbook was to find the closed form solution for deflection, rotation, moment, shear and slope. The author studied different cases of loadings and boundary conditions. The loadings were concentrated load and moment at mid span, two concentrated loads and moments at beam ends and partial and full uniform load. The boundary conditions were simply supported, fixed and free end beams. The author also solved cantilever beam subjected to uniform and concentrated load. Infinite beam subjected to uniform and triangular loads, and concentrated load, concentrated moment. All solved beam cases was assumed to rest on elastic foundation. Other variation parameters such as beam flexural rigidity and elastic properties of the foundation. Two types of elastic foundation were considered, the first type of foundation was Winkler model and the second type was furnished by elastic solid which, in contrast to the first type, represents the case of complete continuity in the supporting medium. In the case of beam subjected to concentrated load resting on continues elastic foundation, the deflection line being a continuous function |(K) of the coordinates absolute difference at the point of load application (ξ) and the place (x), where the deflection was given:

$$w = K\,(x-\xi) \tag{4}$$

If a foundation, characterized by this function, was subjected to any distributed pressure p(ξ)along the length of the beam, the deflection line of the surface of the foundation was given as follows:

$$w = \int_0^l p(\xi) K\,(x-\xi)\, d\xi \tag{5}$$

If the loading on the beam was denoted by q(x), the differential equation of bending of beams was given as follows:

$$\frac{d^4x}{dw^4} = c\,[q(x) - p(x)] \tag{6}$$

where

c: constant

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By combining the equation (5) and equation (6), gives:

$$K(x-\xi)=e^{-c(x-\xi)} \tag{7}$$

The last equation was represent an integro-differential equation which was employed by K. Wieghardt. Wieghardt was the first researcher to investigate the problem of continuous elastic foundation and obtained solutions for finite length beam under definite conditions of loading. The type of foundations corresponding to researcher was not identical with homogenous and isotropic elastic solid, since in the latter has been.

$$K(x-\xi)=C_1\log(x-\xi)+C_2 \tag{8}$$

Terzaghi (1955) presented a method to determine the modulus of sub-grade reaction (Kz) for (sand, clays) soils, based on results of plate bearing test. The author stated that the determination of sub-grade reaction (Kz) depending on hypothesis:

a) The ratio of pressure (p) to the displacement (w) is called (Kz) where the displacement w is not related to the pressure (p).

b) For any point on the surface, the value of (Kz) are constant on the contact surface.

Terzaghi concluded that this hypothesis, is valid for load less than (0.5) of the ultimate bearing pressure (as well as to the value of K_z is dependent on the dimensions of the contact area, the soil properties, depth of soil, stiffness of the beam, length of the beam, and the major effect is found for the distribution of the loading on the beam).

Vesic (1961) developed a formula for ($K_{z)}$ as a function of the soil modulus of elasticity (E_s), soil Poisson ratio, the stiffness and length of the beam. Many engineers conduct a simple load test on the surface of the soil called plate bearing test, to obtain some range of values of K_z. Since the foundation soil in the Winkler model is described by vertical springs only, they carry the vertical loads without any common interaction. In other words, these springs are unable to represent the shear stresses in the subsoil. Shear stresses are developed in the subsoil if the vertical displacements between adjacent springs vary. One can see that if a uniform beam with free ends carries a uniformly distributed load, the solution obtained from the Winkler model is a rigid body displacement. In other words, the calculation will indicate that there is no bending moment in the beam. In reality, there will be some edge shear forces resisting the deformation of the soil which makes the beam to deform in a curved deflected shape, unless the beam is infinitely rigid.

Dodge (1964) summarized a simplified derivation of the basic equations of beams on elastic foundations. The influence functions represent relationship between stiffness beam and stiffness foundation for beams of constant elastic properties and modulus of foundation were obtained. The author was derived expressions to find the deflection, slope, shear and moment for finite beam, infinite beam subjected to line load and moment rest on elastic foundation. The various conditions of loading were applied in the cases. The equations are reduced to formulas containing dimensionless function. Deflection, slope, moment, and shear functions due to unit load or moment were evaluated. The function are tabulated for beams of finite length and plotted for beams of semi-infinite length. The analysis is reduced to the simple evaluation of a one –term algebraic expression by sub-situation the actual (P, M, E, I, b,y,ý, S) the symbols refer to line load, moment, modulus of beam, moment of inertia, width of beam, deflection, slope and shear respectively . For more refined work, the tabulated and plotted data can be used for the construction of influence lines. A higher degree of indeterminacy in structures resting on elastic foundations, introduced by the presence of rigid supports, hinged supports, etc., may be solved by use of functions in equation of continuity. Values of deflections, slopes, moment, and shears found in beams on elastic foundations by any analysis method may be substituted in formulas containing beams functions from which the functions can be readily determined. Application of equations and beam functions in theoretical analysis is illustrated. The basis of comparing beams of finite lengths with semi-infinite and infinite lengths is established. The method for evaluating functions for beams of variable elastic properties and modulus of foundation is explained. Tables and graphs of functions are presented. The tabulated data gives values for the construction of influence lines.

Bowles (1974) developed program technique in the finite element analysis to solve the problem of soil-structure interaction. The programs were coded in Fortran language. Also the author develops and presents new methods for analysis ring foundation and beam on elastic foundation. The methods was calculated with test data and is compared with design procedure. To represent the elastic foundation, Winkler model was used. Several boundary conditions can be used easily. The book discussed continuous footings i.e. two line column on footings, as beam on elastic foundation has been more attention. Two factors justify this trend. First, it is shown that results provide a more realistic distribution of longitudinal bending moment in the member and second the engineers currently are tending to use more mathematical and refined approaches to the solution of engineering problem. Also the author was found another reason for using the beam on elastic foundation, the vertical of subgrade reaction is not enough in the practical range of the modulus of subgrade reaction K_z and footing flexural rigidity EI.

$$K_Z=SF*40q_a \tag{9}$$

where
Sf safety factor an q_a allowable soil pressure

The use of equation (9) has been found to give the consistently reliable and reasonable values of both computed soil pressure and deflections to compare with the given allowable soil pressure. Also, the author used the finite difference and finite element to analyze the problem of mat design. However, the finite difference method has several disadvantage of which the principles ones are as follows:

a) It is difficult to correct for negative deflections i.e. eliminating the springs when footings tends to separate from the soil foundation.
b) It is difficult to code program to generate a general coefficient finite differences matrix.
c) It is difficult to account the nonlinear soil deformation and different load conditions.
d) The finite element method basic solution computer program structure presented by the Bowles is somewhat similar to the finite difference method but eliminates the three major difficulties just cited.

Beaufaitand Hoadley (1980) solved the problem of thin beam resting on nonlinear Winkler elastic foundation with simple support. For solving the differential equation of beam resting on nonlinear foundation, the mid difference method was used. The mid difference method is numerical method used higher order equations. To formulate the concept of mid difference method, replacing the higher order by the first order derivative. The nonlinear problem was solved using the iterative approach. A weighted averaging is used to accelerate the solution. The analysis method is written through a computer program assuming the elastic foundation on Winkler spring to be effective only when it'sin compression. The author investigated two cases of foundation nonlinearity. Two span continuous beam on linear elastic foundation, two span continuous beam on nonlinear elastic foundation. The method has the advantage over other difference methods that might be used to solve this type of problem in that segments of varying length. The difference method for analyzing beam structures were used without complicating the analysis and that few special operators, or equations, are needed to handle difference discontinuities. The obvious disadvantage was the number of equations that must be solved. The technique proposed in this research is very simple and can be organized to solve other types of problems that can be described with the nth order ordinary differential equation.

Yankelevsky and Eisenberger (1985) derived exact stiffness matrix for beam element on Winkler elastic foundation. To represent the continuous part of the beam, a single element is used. For modeling a typical problem, a few number of elements are sufficient. Also, the finite differences are used to solve the problem numerically. The analysis result was compared with the exact solution. The number of mesh were increased to make the numerical solutions results converged to the exact once. Such numerical techniques can easily be extended to deal with the nonlinear foundation problems. The developed computer program takes into account the variation of subgrade reaction and cross sectional dimensions, in assembling the stiffness matrix. The authors studied three case studies. The first case study was free beam resting on springs and loaded with uniform, and concentrated load. The second was partially loaded free end beam on elastic supports. The third case was simply supported beam under concentrated load at mid span. The first and the second case study was analyzed previously by Hetenyi (1946)is solved by author using the program, where beam divided into two elements. For third case the problem was solved by Moher using beam foundation contact stiffness matrix. To find the exact solution, four segments were required. Its found that the results in three cases are in good agreement with the exact solutions.

Matsuda and Sakiyama(1986) developed method for solving the problem of beams on non-uniform elastic foundation. By converting the differential equations into integral equations and then using the numerical integrations, the solutions were found. It is noted that the beams with load cases and boundary condition can be effectively analyzed. To solve the problem, the method is developed as shown in Figure-3. Firstly, the authors studied the cases of beam on uniform elastic foundation, uniform load on Hinged-hinged beam, Fixed-fixed beam, partially uniform Free-free beam supported on (uniform, non uniform) elastic foundation. At first, the beam on uniform elastic foundation was investigated to check the convergence and accuracy of numerical solution. The presented method was employed to the beams with different load cases and boundary conditions. The convergence of the numerical solution was examined by checking the obtained results with the exact solution. Secondly, the authors studied two cases of beam on no uniform elastic foundation. Fixed free beams subjected to triangular distributed loading above the full span, free beam subjected to symmetrically placed uniformly distributed loading. The sub-grade (K_z) varies according to a linear law;

$$K_z = K_A - cx \qquad (10)$$

where K_A and c are constants.

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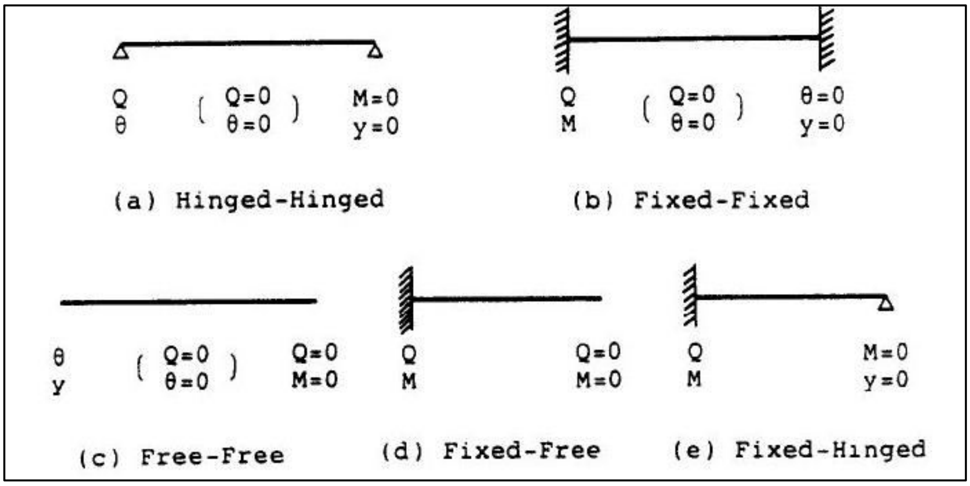

Figure-3. Integral constants and boundary conditions.

Yankelevsky *et al.* (1988) analyzed the problem of beams on nonlinear Winkler foundations. For supported beam on nonlinear elastic foundation, an iterative method based on piecewise linear exact solution was formulated. In continuous beams, the iterative method was coded in computer program to find the deflection, rotation, shear and moment for the beam resting on nonlinear foundation. The authors studied two cases, first continuous prismatic beam under three different uniform loads and four rigid supports, and the second case same the beam under three different concentrated loads and three rigid supports were used for supporting the beam. The solution accuracy was controlled through convergence criteria. The used iterative method converges to the exact solution. For higher values of loads, the foundation behavior becomes nonlinear and large bending happens in the beam. The deflections along the beam were determined after checking the foundation springs to be in compression or tension. When the foundation is in tension the beam regions under consideration are identified. The tensile branch of the deflection curve is specified for these regions and identified for the next iteration.

Al-Jubory (1992)studied the problem of deep beam under static load conditions including the effects of externally moments. Full expression of the basic equation for deep beamwas derived. The frictional and compressional components were modeled by the author. The compressional component was modeled using Winkler model. The friction in foundation is represented by Winker model, Coulomb model and constant friction model. For solving, the basic equations of deep beam, various semi-analytical and numerical techniques such as Fourierseries, the finite element and finite difference were used.Fourier series method is limited to solve directly the case of beams with simply supported boundary conditions. Extensive finite-difference and finite- element formulation were developed to solve various problems. In both Fourier series and finite-difference methods, the formulations are based on the basic equations. While the formulation in finite element technique is based on the adopted displacement functions for deriving the stiffness matrices for the structural elements (beam) and the stiffness matrices for the elastic foundations. In the finite element, the element type interface or (linkage) element connecting the main structure to the foundation were used in the analysis. This element had the property of transferring displacements and stresses from the structure to the foundation. The main deficiency was that those elements occupy spaces (although small in certain cases) between the structure and the foundation. The author studied two case studies. The first case study was simply supported beam on an elastic foundation with Winkler friction and the second case was the clamped-edge beam resting on an elastic foundation with Winkler friction. The first case study was analyzed previously by Fourier series and both numerical methods. For second case the problem was solved by the finite-difference and the finite- element. The results from the three methods (finite difference, finite element and Fourier series) were compared together to check the analysis result. Good agreements were obtained between them. The results show that the three methods are almost identical for smaller depth of beams, obvious difference will appear. The significance of these differences depends on the type of problem. Thus, the shearing forces are more effective in deep beams. Moreover, the Fourier series solution, when applicable, gives stable results while the finite difference or finite element solution depend on the inversion of the coefficient matrices of rather large number of algebraic equations and accordingly the stability of the results may be in correct for small depths of beams.

Onu (2000) derived a formula for modeling beams stiffness matrix which does not depend on beam mesh for numerical model. The stiffness matrix was based on the governing differential equation of beams on elastic foundation that took into account the shear deformation influence. A new stiffness matrix which is free of meshing was obtained considering the shear deformation effect. The authors studied two case studies. The first case study was a free ends finite beam on elastic foundations loaded at ends was utilized for validating the shear stiffness matrix. The second case was the problem of foundation-structure interface for a multi-story building rest on a two-parameter elastic foundation. From this study, two conclusions were obtained. Firstly, the result from the first case showed a new formulation which lead to a reference solution for the results obtained from the displacement field method and secondly the foundation-frame interface example show that, the shear deformation must be considered for beams under static loading.

Yin (2000) introduced a method to solve free ends reinforced concrete (RC) beams resting on elastic foundations subjected to partial pressure of loading. In this method, the particular solution of elastic foundation beam at any point was traced. The deformation shapes, maximum moments and maximum shears of the beam were obtained as a closed form solutions. Fourier cosine series method was used to examine the effect of changing different parameters such as subgrade stiffness (geosynthetic shearing modulus and tension modulus) and loads location.

The pressure q may bea functionof$x$, that is:

$$q = f(x) \text{ for } 0 < x < L \tag{11}$$

where f(x) can be expressed in a Fourier cosine series

$$q(x)= A_o + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}\right) x \tag{12}$$

where:

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx \tag{13}$$

$$A_n = \frac{1}{L}\int_0^L f(x) \cos\left(\frac{n\pi}{L}\right) x \, dx \tag{14}$$

It was concluded that the maximum deflection and maximum tension can be decreased to 17.6% and 9.03% respectively, in comparison with beams that geosynthetic shearing stiffness was not considered. Observation have revealed that deformation shape of them beam showed a better response as modulus of subgrade reaction increased. It was also found that the maximum bending moment, maximum shearing force and tension force increased as modulus of subgrade reaction increased. For example K_z increased from (5,000 kN/m^3- 50,000 kN/m^3), the max. deflection was decreased by 18.2%, but the max. tension was increased by 22.8%.

Husain and Al-Sadder (2000) developed exact secant stiffness matrices for Timoshenko beam on elastic Winkler foundation by using the basic principle approach for matrix displacement analysis. Effect of transverse shear on deflection was taken into account in constructing the uncoupled differential equations. Analytical functions of the complex variable method were used in the solution of uncoupled differential equations. Using this exact element, effects of transverse shear on internal forces and deflection were obtained. In the generation of the element, Winkler hypothesis was assumed, the deflections due to transverse shearing forces were affected as well by the sub-grade reaction K_z. Numerical cases were given to demonstrate the transverse shear effect in beam on an elastic foundations. The authors studied two case studies. The first case study was simply supported beam on elastic foundation under a concentrated load was used to check the accuracy and the validity of the exact secant stiffness matrix and the second case was a beam on elastic foundation under two concentrated load and two concentrated moments at the free ends was used to demonstrate the effect of transverse shear deformation on deflections and internal forces for different value of span to depth ratios and modulus of subgrade reaction. The author has been found the percentage increased in maximum deflection and percentage decreased in maximum bending moment due to transverse shear effect become more significant as the modulus of subgrade reaction increases, accordingly the transverse shear effect must be considered in elastic settlement, the percentage decrease in maximum bending moment was usually less than the percentage increase in maximum deflection (on the safe side if neglected). Also, the maximum percentage increase in maximum deflection was in the range of (3-7)% and this range may increase as the shear correction factor increased (as in inverted tee section where shear correction factor=2). The results of this exact analysis showed that deflection due to shear was influenced for beam will small span-depth ratios values of foundation modulus that effect of transverse shear must be taken into account in elastic settlement calculations.

Guo and Weitsman (2002)developed analytical formulas, to calculate the response (deflection, rotation, moment and reaction) of beams on non-uniform elastic formulation. Winkler model was used to model the no uniform foundation with the modulus of sub-grade is $K_z = K_z(x)$. To solve the problem, Green's function was used. To find the rotation and deflection at location(x) for a simply supported beam subjected to a concentrated load at distance (ξ) from the right beam end.

The used Green's functions for deflection and rotation at location (x) of simply supported beam are given bellow:

$$Gp(x, \xi)= \begin{cases} \frac{1}{6\,EI}\left(\frac{1}{2} - \xi\right)\left(\frac{1}{2}+x\right)\left[\frac{l^2}{2} + l\xi - \xi^2 - lx - x^2 + 6\rho^2\right] for -\frac{1}{2} \le x \le \xi \\ \frac{1}{6\,EI}\left(\frac{1}{2} - x\right)\left(\frac{1}{2}+\xi\right)\left[\frac{l^2}{2} + l\xi - \xi^2 - l\xi - \xi^2 + 6\rho^2\right] for \xi \le x \le \xi - \frac{1}{2} \end{cases} \tag{15}$$

where (ρ, E, I) shear deformation parameter, modulus of elasticity, moment of inertia respectively

$$\rho^2 = \frac{EI}{Ko\,GA} \tag{16}$$

where the

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Ko (shear correction factor) $=\frac{5}{6}$ (17)

(A, G) cross sectional area and shear modulus of the beam respectively

$$G_p(x,\xi)=\begin{cases} -\frac{1}{6\,EI}\left(\frac{1}{2}+x\right)[x^2+lx-3l\xi-lx-3\xi^2-6\rho^2]\,for\; -\frac{1}{2}\le x\le \xi \\ -\frac{1}{6\,EI}\left(x-\frac{1}{2}\right)[x^2-lx-3l\xi+3\xi^2-6\rho^2]\,for\quad \xi\le x\le \xi\frac{1}{2} \end{cases} \tag{18}$$

Note: $G_{M=}-\partial G_P/\partial\xi$

The authors studied three case studies of simply supported beams with length of 5 m and under concentrated load at mid length and resting on variable value of subgrade reactions.

Case1: the sub-grade reaction (K_{z1})=12MPafor x<0.08m and for x<5m,the sub-grade reaction (K_{z2})=25 MPa

Case 2: the sub-grade reaction (K_{z1})=12MPafor x<0.08m and for x <0.2, the sub-grade reaction(K_{z2})=400/(x-0.08), the sub-grade reaction (K_{z3})=25 MPa, for x<5m

Case 3: the modulus of sub-grade reaction variation between the left end and the right end of the beam, where the K_{z1}=25 MPa for x=−0.05 and K_{z2}=0 for x=0.05

For the third case, the distributed reaction p(x) and moment M(x) results were compared with simply supported beam resting on uniform foundation, (uniform of modulus sub-grade reaction (K_z =15 MPa) are shown in Figure-4andFigure-5.Also, the results of for first and second caseareshowninFigure-6andFigure-7.

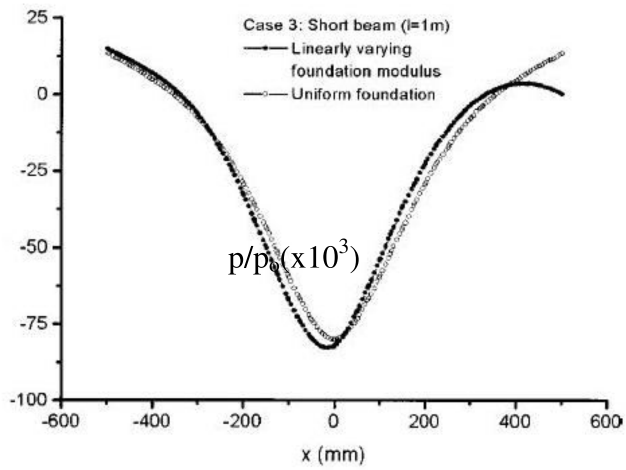

Figure-4. Distributed reaction p(x) scaled by p_o=P/b (point load (P)= -20 kN, width of the beam b=25 mm).

Also, to find the rotation and deflection at location (ξ) for a simply supported beam under moment at location ξ

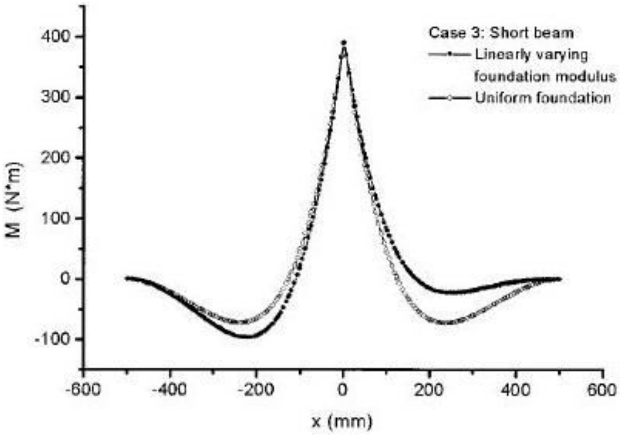

Figure-5. Bending moment M(x) for various x.

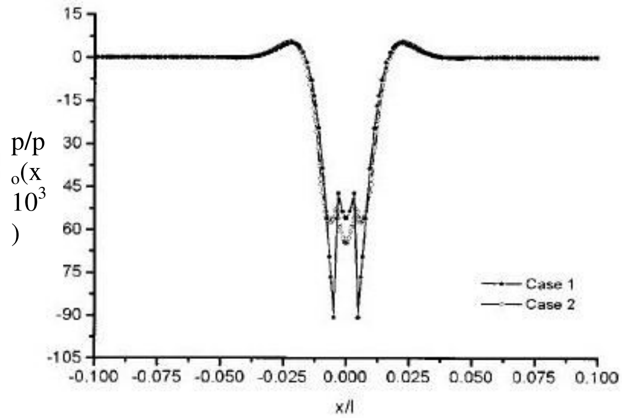

Figure-6. Distributed reaction Q(x) scaled by p_o=P/b (point load (P)= -20 kN, width of the beam b=25 mm).

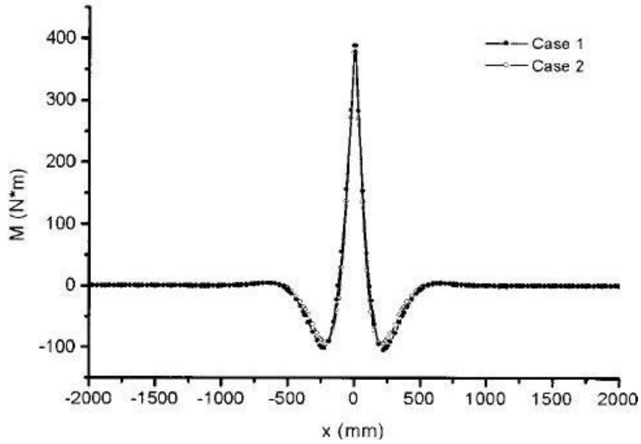

Figure-7. Bending moment M(x) for various x.

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The advantage of this method can be applied to soils and beams of variable properties and sizes by simply adjusting these input variables.

Al-Azzawi and Al-Ani (2004) solved the problem of thin beams on Winkler foundations with both tangential and normal friction resistances. The research was an attempt to analyze flexible beams on Winkler elastic foundation by the use of finite-differences method. The authors studied two case studies. The first one, simply supported beam under uniform load including or excluding the friction effect. The second case was of free beam under two concentrated loads including or excluding the friction effect. For two cases, the problem was solved by exact and finite difference solutions. It was found that the friction in beam foundation interface to be small on moment, deflection and shear which decreased by increasing the horizontal subgrade reaction coefficient. The used finite difference mesh was (Δx=0.1). A better result could be attained by increasing the number the nodes (decreasing Δx). The result from the exact and finite difference solutions were in good agreement and checked the accuracy of the method used in that research. Finally, the researcher has noticed that the effect of variable vertical k_z on moment, deflection, shear was obvious.

Al-Musawi (2005) solved the problem of beams resting on elastic foundations using different types of finite elements, (one, two, and three dimensional iso parametric finite elements), to model the beam. Fortran-coded program was developed for the analysis. The author studied four case studies. The first and second cases were, the simply supported beams and the fixed-end beams; the loads on beams for these two cases were uniformly distributed. The third case considered free-end beam under two concentrated loads at both ends. The fourth case considered, the cantilever beam under single concentrated load at the free end. The author examined three parameters in the study that directly affect the moments, deflections, and sharing forces. The parameters were the horizontal and vertical sub-grade reactions, and the depth of beams. The findings of the author study were compared with previous studies numerical, analytical and exact results. Such compatible agreement indicates the efficiency of the used method in that study. It has been also found that the maximum difference in the central deflection between the finite difference method and the author study was (0.657%) and the variance between the author study and the exact solution was (0.326%).Finally, the researcher has noticed that the convergence of the results depends on the number and the type of elements used in solving the problem in question.

Abbas (2007) solved the problem of deep beams supported on two-parameter elastic foundations. These parameters included the vertical and horizontal membrane resistances. To analyzed deep beams, Abbas further used a Fortran-coded finite difference program. The author analyzed four case studies. The first and second cases were studied, simply supported beam resting on one-parameter elastic foundation. However, in the first case, the beam load was uniformly distributed whereas in the second case, the load was concentrated at mid span. In both two cases, membrane force (*T*) was assumed equal to zero. The third and fourth cases were analyzed free ends deep beam resting on an elastic foundation under single concentrated at mid span and two concentrated loads at the free ends [Al-Azzawi and Abbas (2010)].The author examined three parameters which directly effect on moments, deflections, and sharing forces. These parameters were the vertical sub-grade reactions, membrane force width and depth of the beam. The findings of author study were compared with previous studies numerical, exact solutions and analytical methods to check the validity of the developed analysis. The average of differences for moments and deflections were (1.6 %), and (3.9%), respectively. In all cases studied before, the membrane force (*T*) increased from (0.0 to10000kN) whereas the deflection decreased. In first case which has a uniform load, the membrane force increased while the percentage of deflection decreased to (78.4%). Finally, the researcher has noticed that the effect of increasing depth of the beam on stress resultants (shear force and bending moments) was lesser than that on deflections. In first case that has a uniform load, when the depth of beam increased from (0.25 to 1.0m), the percentages were (96.3%), (31%), and (15.9%) for deflection, shear and moment, respectively. Also, the effects of width of the beam, vertical sub-grade (K_z) on the behavior of the beam was obvious. For simply supported beam, when K_z increased from (0 kN/m^3- 3,000 kN/m^3) and width of the beam increased from (0.25 m- 1.0 m), the max. deflection was decreased by (4.2%) and (72%), respectively.

Al-Azzawi (2008) presented a numerical solution to calculate the response (deflection, moment and shear) of deep beams on non-homogeneous elastic compressional and frictional Winkler foundations. To proceed with the study, the author used finite element method for solving problem of deep beams on both horizontal (K_x) and vertical (K_z)sub-grade modulus. To solve the problem of one dimensional beam on an elastic foundation, the program offered by Owen and Hinton was improved. The modified program was coded with Fortran_77 computer language. The results obtained from author study were compared with analytical solution using Green's function formulation for the cases imply supported beam resting on homogeneous and non-homogeneous vertical subgrade reaction under a concentrated load. The author examined two parameters in the study that directly affect the moments, deflections, and sharing forces. The parameters were horizontal (K_x) and vertical (K_z) sub-grade reactions, and depth of the beam. The result obtained from the finite element and Green's functions were in good agreement. Finally, the researcher has noticed that the effect of non-uniformity of frictional restraint at the foundation-beam system was found to be smaller on the behavior (moment, deflection, and shear) than the effect of non-uniformity of compressional restraint. Also, the researcher has noticed that the effect depth of the beam on the behavior of beams was found to be similar for uniform and non-uniform subgrade reactions cases.

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Al-Azzawi *et al*. (2010) solved the problem of deep beams supported on nonlinear and linear elastic media. In this study, to solve the problem and analyzed deep beams, the finite elements ANSYS program was applied using (SOLID65) to model the beam and the foundation can be represented by linear spring elements (COMBIN14) and nonlinear spring element (COMBIN39). To represent beam, three-dimensional eight node brick elements had been used. The authors analyzed three case studies. The first and second cases were studied, the simply supported beams and the free-end beams. However, in the first case, the beam load was uniformly distributed whereas in the second case, the load was single concentrated at the end. The authors studied in the third case, cantilever deep beam under a concentrated load. The findings of author study were compared with previous numerical studies to check the validity of the developed analysis. The (ANSYS-v11) program that was used in authors study showed excellent agreement for deflections of about (4.8%) and good agreement for moments of about (25%).The author examined three parameters in the study that directly affect the moments, deflections, and sharing forces. The parameters were the vertical sub-grade kz, mesh size, and compressive strength of concrete - to study their effects on the beam behavior It was found that the maximum deflection was increased by (2.5%) when the foundation was modeled as a beam on nonlinear Winkler model rather than linear Winkler model because of the accumulative deflection with load increments and the reduced modulus subgrade reaction. Finally, the researcher has noticed that the effect of varying the modulus of subgrade reaction on deflection for linear Winkler model was found to decrease the deflection by (9.8%) for K_z from (0 to 10000 kN/m^3). Also, the effect of large deflections on deep beams was found to be small on results about (0.11%).

Al-Azzawi (2010) solved the problem of thin beam resting on nonlinear elastic media. The results obtained of plate bearing test in Iraqi soils and calculated by a tangent method were used in this analysis. To solve the nonlinearity of foundation, an iterative method was applied. The author analyzed two case studies. The first and second cases were studied, simply supported beam resting elastic foundation. However, in the first case, the beam load was uniformly distributed whereas in the second case, the load was concentrated at mid span. The author solved these cases by many methods depending on type of support and applied loading such as Fourier series method, (finite difference, finite element) techniques and exact solutions. The results obtained from the analysis shows closer values for shear under higher values of applied loading on the beam but rather different values for bending moment and deflection. Finally, the researcher has noticed that the effect depth of the beam was significant on moment, deflection and shear with percentages of (86.6%), (33.36%) and (55.6%), respectively.

Al- Azzawi and Theeban (2010) conducted a study on large deflection of deep beams resting on Winkler foundation. For solving the problem of deep and thin beams, a finite difference technique was applied. To proceed with the study, the author used the element SHELL of 43 in the finite element ANSYS 5.4 program. A full Newton-Raphson option was used in finite element analysis. To solve the nonlinearity of foundation, Newton-Raphson method and an incremental load approach were applied. The results arrived from finite differences were compared with that obtained from the finite element method, (ANSYS program) for checking the analysis results. The authors studied four cases; the first case was thin and deep cantilever beam under a concentrated load at free end**.** The author studied in the second case thin and deep cantilever beam under a uniformly distributed load. The third and fourth cases were simply supported beams and the fixed-end beams. However, in the third case, the load was concentrated at mid span whereas the in the fourth case, the load was uniform. The author examined three parameters which directly effect on moments, deflections, and sharing forces. These parameters were the vertical sub-grade reactions, width of beam and the ratio of depth to the length of the beam. The convergence of the results depends on the number of nodes (from 5 to 100 nodes) adopted in the finite-difference mesh used for beam modeling. The results have shown that the maximum moment, deflection and shear for all cases increased and converged to the exact values. Two approaches, numerical and exact, were used to compare and check the solutions obtained from the finite difference. The maximum difference in deflection at mid span between the finite difference and the finite element was (0.79%). Moreover, the difference between the present study and the exact solution was (0.86%). Finally, the researcher has noticed that the effects of width of the beam, vertical sub-grade (K_z) on the behavior of the beam was more obvious as the depth increases. In second case that has a uniform load, when the K_z increased from (0 -6000 kN/m^3), the max. Moment, deflection, and shear were decreased by (3.81%), (3.7%), and (3.2%), respectively. Also, the researcher has noticed that the effect of increasing ratio of depth to the length of the beam from (0.2-1.0), the max. deflection, moment and shear were decreased by(96.4%),(9.8%), and (9.3%), respectively.

Al- Azzawi and Al-Obaidie (2010) investigated the nonlinear analysis of reinforced concrete beams resting on linear Winkler foundation. Material nonlinearities due to cracking of concrete, plastic flow, crushing of concrete and yielding of reinforcement were considered. A modified finite element computer program was developed for solving the problem. The Winkler model for both tangential and normal were added to the original program (DARCEF) Dynamic Analysis of Reinforced Concrete Beams on Elastic Foundations) to represent the foundation and subjected to different types of loading. The authors investigated two case studies. The first and second cases studied, simply supported beam resting elastic foundation. However, in the first case, the beam load was uniformly distributed whereas in the second case, the load was partial uniformly distributed load over length (100mm) at mid span. The findings of author study were compared with exact solutions and previous numerical studies to check the validity of the developed analysis. In first case, the problem analyzed by Al-Jubory (1992) and also by Al-Musawi (2005) was solved using the developed computer program whereas in the second case, the problem was analyzed by Yin (2000). The obtained solutions from authors study were in good agreement with the previous studies results. Maximum percentage of difference in deflection was 15%. The authors examined two parameters which directly effect on behavior of beam. These parameters were cross sectional dimensions of the beam and tangential (K_x), normal (K_z) sub-grade. In previous two cases, when the width and depth of the beam increased from (150 mm -450 mm), and (300 mm -900 mm),the maximum deflection was decreased by (32.2%), and (16.6%), respectively. Also, when (K_x) and (K_z) increased from (0.128 N/mm3 to 1.024 N/mm3), the maximum deflection was decreased by (9%) and (51%), respectively.

Al- Azzawi (2011) solved the problem of deep beams supported on nonlinear or linear Winkler foundations with both frictional and compressional resistances. In this study, to solve the problem and analyzed deep beams, the finite elements method was utilized through using elastic springs and two dimensional plane stress isoparametric element to model the foundation and deep beam, respectively. Also, to solve the basic equation of deep beam, two Fortran-77 programs were formulated to deal with various boundary conditions such as simply supported and free ends. To obtain rotations and deflections at each node, the differential equations were converted (w and ψ) into finite differences. Also, to obtain a number of simultaneous algebraic equations, assembling the difference equations for all nodes after applying the boundary conditions. Then using Gauss Jordan to solve these equations. The author studied two case studies. The first and second cases deals with simply supported beam resting elastic foundation. However, in the first case, the beam load was uniformly distributed and foundation was linear whereas in the second case, the load was single concentrated at mid span and foundation was treated as nonlinear. The results obtained from the author study finite difference and finite element was compared with previous studies numerical, analytical and exact solution results. Such compatible agreement indicates the efficiency of the used method in that study. It has been found that the maximum difference for moment, deflection and shear between methods was (3%). The author examined three parameters which directly effect on behavior of beam; these parameters were horizontal (K_x), vertical (K_z) sub-grade and depth of the beam. The researcher has noticed that the effect depth of the beam and (K_z) was obvious on the behavior of beam. Also, it was found the effect of (K_x) was lesser than the effect of (K_z) on the results of moment, shear and deflection. The researcher has noticed that the effect depth of the beam on the behavior of beams was found to be approximately similar for moment and shear in case of linear and nonlinear subgrade reactions. Finally, the researcher has noticed that the effect depth of the beam was significant on moment, deflection and shear with percentages of (94.6%), (50.6%) and (66.67%), respectively.

Zhan (2012) solved the problem of beams resting on elastic media. In that study, to solve the problem and analyzed beams, the finite elements ABAQUS (6.10) program was utilized through using spring and plate (shell) element to model the foundation and beam, respectively. The author studied two case studies. The first case was free ends beam subjected to uniform load and concentrated force at mid span. The author studied in the second case free ends beam, length of the beam (10 m) subjected to force at (1.0 m) and moment at (4.0 m) from left hand whereas the load was uniformly distributed at distance (5.0 m) from right hand end. The findings of the author study finite element was compared with previous analytical studies. The obtained solutions from authors study were in good agreement with the previous analytical studies results. It has been found that the maximum percentage of difference in deflection and moment between the finite element and analytical solutions were (0.31 %) and (0.75%), respectively.

Al-Shaarbaf *et al*. (2012) solved the problem of linear deep beams supported on Winkler elastic foundations. In this study, to solve the problem and analyzed deep beams, the finite elements method was utilized through using elastic springs and one dimensional isoparametric element to model the foundation and deep beam, respectively. Fortran-coded program was formulated in that analysis. The authors studied the case of simply supported beam under uniform load. The findings of author study were compared with exact solutions and previous numerical studies to check the validity of the developed analysis. It was found that the maximum difference in central deflection between finite difference method used by other researchers and the present study was 1% and the difference with the exact solution was 0.98%. The authors examined three parameters which directly effects the behavior of beam, these parameters were horizontal (K_x), vertical (K_z) sub-grade and depth of the beam. The researchers has noticed that the effect of frictional restraint at the foundation-beam system was found to be smaller on the behavior (moment, deflection, and shear) of the beam with maximum percentage difference of (1%) than the effect of compressional restraint with maximum percentage difference of (67%). Finally, the researchers have noticed that the effect depth of the beam on stress resultant was found to be lesser (74%) than the effect on deflection (93%).

Jang *et al*. (2013) solved the problem of an infinite beam on a nonlinear elastic foundation. In this study, to represent and solve the nonlinearity of foundation, one way spring model and (an iterative method, Green's function technique) were used, respectively. One way spring model (one curvature for load -settlement relationship) was used for analyzing the foundation nonlinearity instead of conventional two way spring model. The findings of author study were compared with numerical experiments to check the validity of the developed analysis. This study assumed a nonlinear

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spring force $f\ (u)$ in Eq. (19) and to check the accuracy of the results converges an exact solutions.

$$f(u)=\begin{cases} K_z.u+N(u), & for\ u>0 \\ 0 & for\ u\leq 0 \end{cases} \quad (19)$$

Where:
$N(u)$a nonlinear part of spring force.

Where the exact solutions compared with numerical experiments. To check the results obtained from iterative method, three cases of exact solutions were used listed in Table-1 and shown in Figure-8. The applied loading is shown in Figure-9.

Table-1. Three cases of the exact solution.

Case	Exact solution $u(x)$
a	e^{-x^2}
b	$\mathrm{Sin}\,x\cdot e^{-x^2}$
c	$\mathrm{Sin}\,x\cdot e^{-x^2/4}$

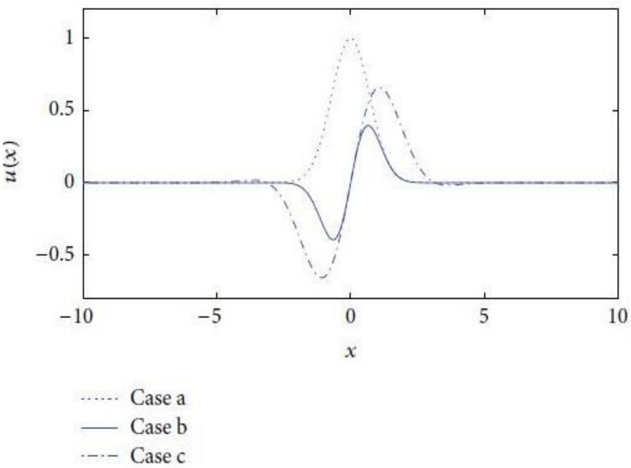

Figure-8. Exact solutions.

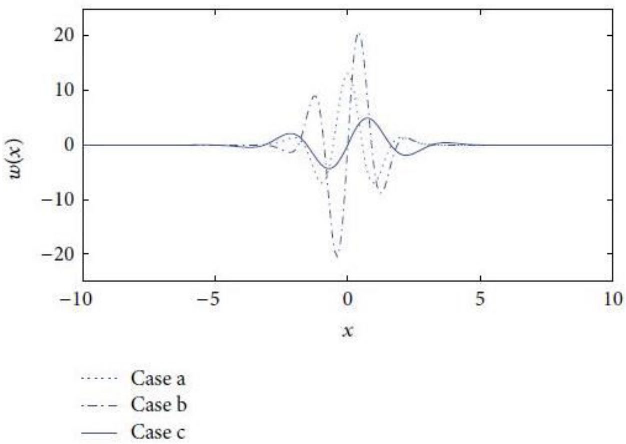

Figure-9. Applied loading conditions corresponding to the exact solutions in Table-1.

To determine high foundation nonlinearity, an iterative method was applied. The results obtained from the exact, numerical solutions were in good agreement as shown in Figure-10.

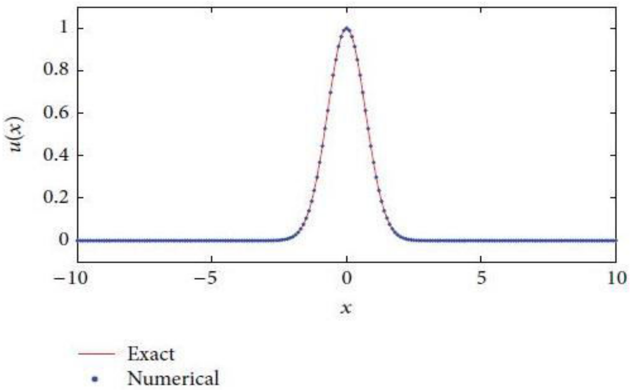

Figure-10. Exact solutions compared to the Numerical solutions.

Tiwari and Kuppa (2014) studied the problem of beam on elastic foundation under static and dynamic loads. The foundation model was resting on Winkler hypothesis. Various methods were described for treating the problem subjected to point load in transverse direction. An example problem was solved using finite element package ANSYS. The authors studied the case of point load acting at center of the simply supported beam (Static case). This case was solved using finite element package ANSYS where the beam was modeled with two equal elements. The results showed the behavior of beams on elastic foundation when subjected to static load as shown inTable-2. This provided a groundwork to study structures on continuous foundations.

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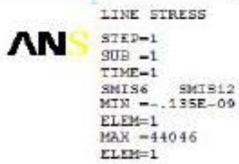
Table-2. Maximum, minimum values under static load.

	Maximum	Minimum
Deflection (m)	-0.001453	-0.001288
Moment (Nm)	44046	0
Shear Force (N)	+50000	-50000



Bogdan (2015) solved the problem of beams resting on elastic foundation. To proceed with the study, the author used finite difference method for solving problem. Matlab-coded program was developed. The authors studied the case of free ends beams subjected to three concentrated forces at different location. The findings of author study were compared with finite elements and previous analytical studies to check the validity of the developed analysis. The obtained solutions from authors study were in good agreement with the finite element and previous analytical studies results. It was found that the maximum difference in moment between finite difference method and the finite element was (0.44%).It has been found that the finite element method solution yields appropriate results for positive moments, whereas finite difference method solution yields a good result for negative bending moments. Finally, the researcher has noticed that the convergence of the results depends on the number of nodes adopted in the finite-difference mesh used for beam modeling. Finite difference solution doesn't yields a result for positive bending moments but by increasing the mesh size, the solution yields a good result for positive bending moments.

Al- Azzawi (2017) conducted a study on shear deformation in hollow beam supported on Winkler foundation. To proceed with the study, the author used finite difference method for solving problem. The author analyzed two case studies. The first and second cases were studied, simply supported beam resting elastic foundation under a uniform load. However, in the first case, the beam was solid whereas in the second case, the beam was hollow. The findings of author study were compared with exact solutions and previous analytical studies to check the validity of the developed analysis. In previous two cases, the problem analyzed by Bowles (1974) analytically. The author examined four parameters which directly effect on moments, deflections, and sharing forces. The study parameters were the vertical sub-grade reactions (K_z), depth of the length, hole dimensions and type of loadings. It has been also found that the maximum difference in the central moment, deflection and shear between the exact solution and the author study were (2.5%), (6%) and (2.4%), respectively. Such compatible agreement indicates the efficiency of the used method in that study. In first case, when the depth of beam increased from (0.4 to 1.2m), the percentages were (36%), (33%), and (94%) for moment, shear and deflection, respectively. Also, when (K_z) increased from (0 kN/m^3- 50000 kN/m^3), the max. deflection was decreased by (16%). The researcher has noticed that the effect of increasing hollow from (0-38%), the max. moment, deflection and shear were decreased by (10%),(21%), and (10%), respectively. Finally, the researchers has noticed that the effect of loading was found to be negligible on shear than the effect of loading on deflection and moment were more obvious.

CONCLUSIONS

Based on previous studies research, the following conclusions can be obtained:

a) Finite element method is the most practical method used in the analysis of beams on elastic foundation due to large applications.

b) It is noticed that the effect beam depth on stress resultant is lesser than the effect on deflection.

c) It is noticed that the effect of beam depth on the behavior of the beam is more significant than beam width.

d) It is noticed that the effect of vertical subgrade reaction on the behavior of the beam is more significant than horizontal subgrade reaction. As the vertical subgrade increased, the maximum moment, deflection, and shear are decreased.

e) It is noticed that the effect of type of loading is found to be negligible on shear than the effect of loading on deflection and moment.

f) It is noticed that the maximum deflection was increased when the foundation was modeled as a beam on no uniform Winkler model rather than uniform Winkler model because of the accumulative deflection with load increments and the reduced modulus subgrade reaction.

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