ANALYTICAL AND PRACTICAL METHODS TO RELATE TIME AND FREQUENCY PARAMETERS OF TRANSFER FUNCTIONS

Diego F. Sendoya-Losada, Johan Julián Molina Mosquera and Jesús D. Quintero-Polanco
Department of Electronic Engineering, Faculty of Engineering, Surcolombiana University, Neiva, Huila, Colombia
E-Mail: diego.sendoya@usco.edu.co

ABSTRACT

In the present work, both analytical and practical methods are employed, in order to find a relation between time and frequency parameters of transfer functions: Overshoot, Phase Margin, Crossover frequency, and Bandwidth frequency. The final aim it to contribute to the design of a novel PID auto-tuner for cases where the existing auto-tuning strategies have a poor performance.

Keywords: auto-tuners, frequency response, PID, time response, transfer function.

1. INTRODUCTION

Despite the numerous existing procedures, much attention has been given in recent years to methods for the design of PID controllers. The main reason is that the controllers are simple, easy to implement and give good performance. Åström and Hägglund introduced a relay based method for the automatic tuning of PID. The relay auto-tuning method is based on the critical point (critical gain and critical frequency) which lies on the negative real axis of the complex plane.

In the present work, both analytical and practical methods are employed, in order to find a relation between time and frequency parameters of transfer functions: Overshoot (OS%), Phase Margin (PM), Crossover frequency (ϖ_c) and Bandwidth frequency (ϖ_{BW}). The final aim it to contribute to the design of a novel PID auto-tuner for cases where the existing auto-tuning strategies have a poor performance.

The paper is structured as follows: First, the analytical procedure and the final graphs obtained for parameters studied are presented. Then, the in-house developed computer aided control system design FRtool is used, in order to find the relation between Overshoot percent (OS%) and Phase Margin (PM), and the results from analytical approach are compared. Furthermore, PID controllers are designed for nine different linear systems, which are typical cases for control and serve as validation of auto-tuning PID methods; a maximal OS% of 20% was taken as parameter design. Then, controllers are designed for a singular OS% specification in two different systems, in order to verify if relations found in past sections are still useful. A final section summarizes the main outcome of these experiments and suggests next steps for research.

2. MATERIALS AND METHODS

The analytical procedure to find the relation between the open-loop and the closed-loop responses for several important parameters, are based on the approximation of the system to a second order in closed-loop. This means that the system has two dominant complex conjugated poles in the left part of the complex plane. This approximation will be evaluated in simulations using Matlab and Simulink platforms.

In order to find relation between the OS% (time response parameter in closed-loop) and PM (frequency response parameter in open-loop), is necessary to use the following open-loop transfer function:

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

Assuming that the magnitude of the transfer function $G(s) = G(j\omega) = 1$, it is possible to find the frequency ϖ, resulting in:

$$\varpi = \varpi_n\sqrt{-2\zeta^2 + \sqrt{4\zeta^2 + 1}}$$

This frequency can now be used in the phase of the transfer function $G(s)$ in order to get a relation between the damping factor (ζ) and the phase (φ) of the transfer function. The phase margin is defined as the difference between this angle and -180º (intersection with the real negative axis).

$$PM = \tan^{-1}\left(\frac{-2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}\right)$$

Giving values $0 < \zeta < 1$, it is possible to get a relation between ζ and PM as depicted in Figure-1.
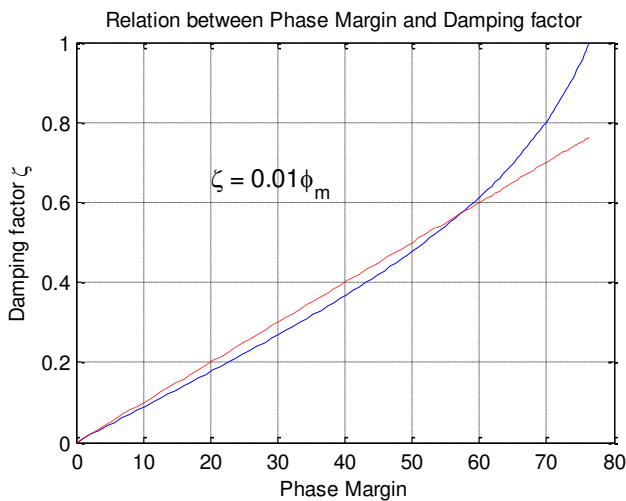
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Figure-1. Relation between PM and damping factor.

Based on the linear approximation of the data obtained from Figure-1, an important relation could be found:

$$\zeta = 0.01 \, PM$$

The final representation of the PM (in open-loop) and the OS% (in closed-loop) is given by the following relation:

$$PM = \tan^{-1}\left(\frac{-2b}{\sqrt{-2b^2 + \sqrt{4b^4 + a^2}}}\right)$$

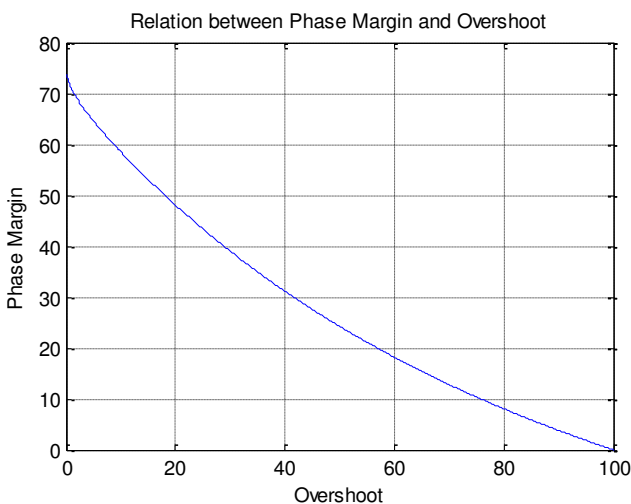with $b = \ln(\%OS/100)$ and $a = \pi^2 + b^2$. This relation is plotted in Figure-2.

Figure-2. Relation between OS% and PM.

It is important to have in mind that these relations were obtained as approximation with a second order system; hence an error will be introduced for other types of systems.

In order to find a relationship between ϖ_{BW} and ϖ_c, the magnitude of the system in closed loop is evaluated at -3dB, in order to get the ϖ_{BW}, obtaining the following equation:

$$G(s) = \frac{\omega_n{}^2}{s(s + 2\zeta\omega_n)}$$

$$T(s) = \frac{G(s)}{1 + G(s)}$$

$$|T(j\omega)| = \frac{\omega_n{}^2}{\sqrt{(\omega_n{}^2 - \omega_{BW}{}^2)^2 + (2\zeta\omega_n\omega_{BW})^2}} = \frac{1}{\sqrt{2}}$$

$$\omega_{BW} = \omega_n\sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

or similarly:

$$\frac{\omega_{BW}}{\omega_n} = \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

This important relation is depicted in Figure-3, for values $0 < \zeta < 1$.
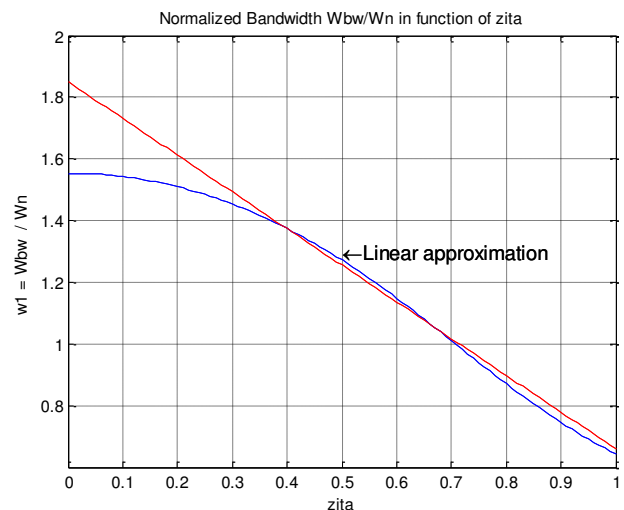

Figure-3. Relation $\frac{\omega_{BW}}{\omega_n}$ vs ζ.

In order to find ϖ_c, the magnitude of the open loop transfer function must be evaluated for 0dB.

$$|G(j\varpi)| = \frac{\varpi_n{}^2}{\sqrt{(-\varpi_c{}^2)^2 + (2\zeta\varpi_n\varpi_c)^2}} = 1$$

$$\varpi_c = \varpi_n\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}$$

$$\varpi_n = \frac{\varpi_c}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}$$

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$$\frac{\omega_{BW}}{\omega_c} = \frac{\sqrt{(1 - \zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}$$

From these results, it is possible to draw the following relation Figure-4:

$$1.32\omega_c \leq \omega_{BW} \leq 1.62\omega_c$$

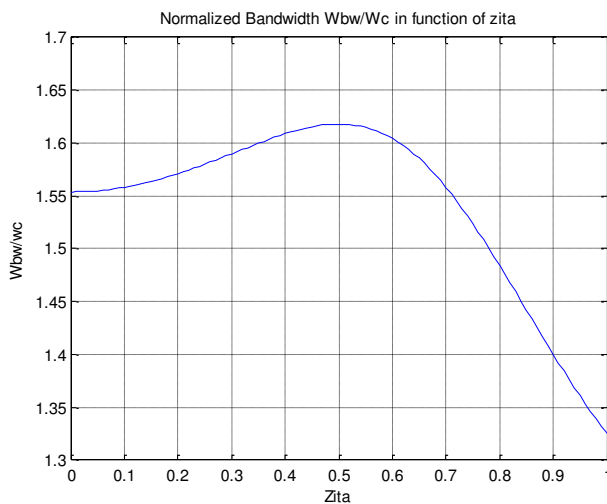$$\frac{\omega_{bw}}{\omega_c} \cong 1.5$$

Figure-4. Relation $\frac{\omega_{BW}}{\omega_c}$ vs ζ.

3. RESULTS AND DISCUSSIONS

In this section Frequency Response toolbox (FRtool) for Matlab® is used. Probably the most important feature of FRtool is the user friendly graphical interface (drag & drop and zoom included). Initially the transfer function of the process is not used, just the measurement of the Phase Margin necessary for different values of OS% are taken.
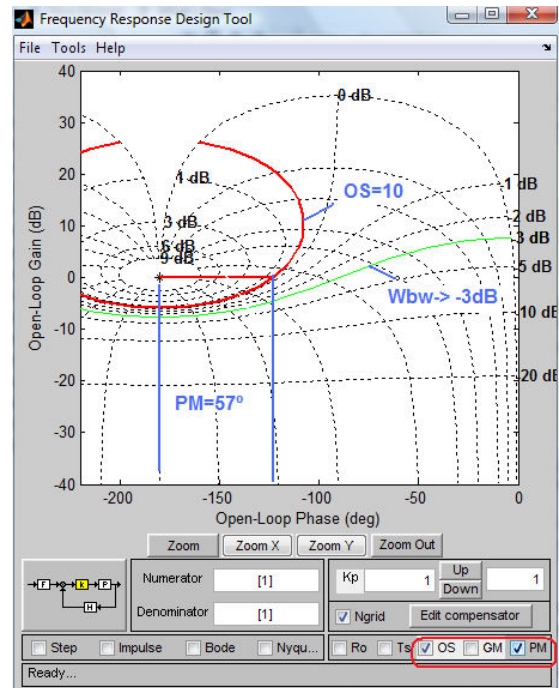

Figure-5. FRtool interface.

In Table-1 the results from analytical approach and CAD design are compared. Figure-6 depicts the results of Table-1, i.e. the relationship between OS% and PM. Analyzing the graph is possible to observe that for both approaches exist an error in the PM for values less than 15% of OS%; but this error is still lower than 6%, and hence acceptable.

$$Error\% = \frac{|PM_{FRTool} - PM_{Analytical}|}{PM_{FRTool}} * 100$$

Table-1. PM obtained using FRtool and analytical graphs.

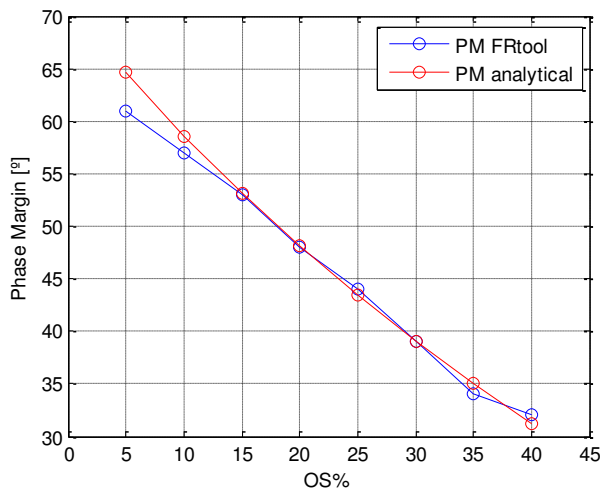| OS%	PM FRtool	PM analytical	Error%
5	61	64.63	5.95
10	57	58.59	2.79
15	53	53.17	0.32
20	48	48.15	0.31
25	44	43.46	1.23
30	39	39.09	0.23
35	34	35.01	2.97
40	32	31.19	2.53

Figure-6. PM vs. OS% analytical and FRtool approach.

Now, nine typical examples for control engineers are proposed, namely those given in Table-2, which are of interest for PID auto-tuners rules. These include high order systems and non-minimum phase models with integrator. In order to use a nomenclature, the following representation will be used:

- **Order system:** 1st, 2nd , 6th (e.g. for first, second or sixth orders)
- **Time delay:**TD (if the system has time delay)
- **Integrator:** Int (if transfer function that represents the system has an integrator)
- **Non minimum phase:**NM (especial system with positives zeros)
- **Under damped:**UD (especial under damped system with low ζ)

Table-2. Transfer function models used in the experiment.

TF	Nomenclature	Transfer function
1	2nd + TD	$G(s) = \dfrac{2e^{-3s}}{(1 + 10s)(1 + 5s)}$
2	2nd +TD	$G(s) = \dfrac{e^{-0.2s}}{(1 + s)^2}$
3	1st + TD+ Int	$G(s) = \dfrac{e^{-0.3s}}{s(s + 1)}$
4	1st + TD+ Int	$G(s) = \dfrac{e^{-0.2s}}{s(s + 1)}$
5	6th	$G(s) = \dfrac{1}{(s + 1)^6}$
6	2nd + Int	$G(s) = \dfrac{32}{s(s + 4)(s + 16)}$
7	1st + Int +TD+ NM	$G(s) = \dfrac{0.25(1 - s)}{s(2s + 1)} e^{-0.2s}$
8	4th + TD	$G(s) = \dfrac{17.7e^{-s}}{(s + 1)(30s + 1)\left(\dfrac{s^2}{9} + \right.}$
9	2nd + UD	$G(s) = \dfrac{0.74}{s^2 + 1.48s + 481.48}$

In order to design the controller, an OS% less than 20% was taken as design specification. The basic (textbook) transfer function of a PID is given in terms of the three gains: proportional, integral and derivative, and its equivalent pole-zero-gain representation:

$$C(s) = K_p \left(1 + \frac{1}{T_i . s} + T_d . s\right) = K \frac{(s - z_1)(s - z_2)}{s}$$

The PID controller was designed using two identical real zeros in order to get the fastest response in closed loop with no overshoot (Figure-7).
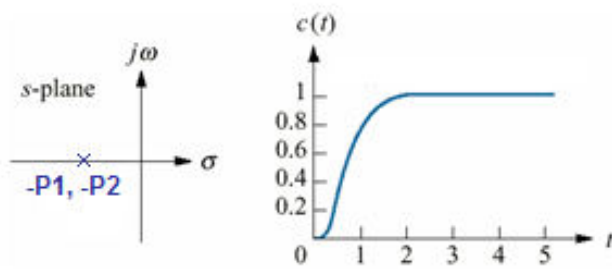
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Figure-7. Closed response of a system with two real poles.

The transfer function of the designed controllers appears summarized in Table-3. An important conclusion is obtained with this experiment, because the relation between crossover and bandwidth frequencies was in average 2, which is closed to theory data in where mentions a relation of 1.6.

$$\frac{\omega_{BW}}{\omega_c} \cong 2$$

Then, PID controllers were designed for different OS% values in two different systems, in order to verify if relations found in past sections are still useful. Using the following models (Transfer functions #3 and #5) was used:

$$G_3(s) = \frac{e^{-0.3s}}{s(s+1)}$$

$$G_5(s) = \frac{1}{(S+1)^6}$$

Table-3. Controller designed and summary of the parameters studied.

TF	Nomenclature	Controller Designed	OS%	Ts	GM	PM	ϖ_{BW}	ϖ_c	$\dfrac{\varpi_{BW}}{\varpi_c}$
1	2nd + TD	$C(s) = 4.9246\dfrac{(s^2 + 0.2808s + 0.01971)}{s}$	15	18.7	8.79	59.4	0.456	0.185	2.46
2	2nd +TD	$C(s) = 2.7548\dfrac{(s^2 + 2s + 1)}{s}$	26.8	2.2	9.1	58.4	6.45	2.74	2.35
3	1st + TD+ Int	$C(s) = 1.9\dfrac{(s^2 + 2.188s + 0.6302)}{s}$	38	6.6	8.675	52	4.25	2.3	1.85
4	1st + TD+ Int	$C(s) = 1.6582\dfrac{(s^2 + 1.279s + 0.4088)}{s}$	23.9	6.01	13.3	60	2.94	1.65	1.78
5	6th	$C(s) = 0.60114\dfrac{(s^2 + 1.271s + 0.4036)}{s}$	18.1	23.1	8.97	51.3	0.46	0.22	2.09
6	2nd + Int	$C(s) = 3.204\dfrac{(s+3)^2}{s}$	28.3	1.16	Inf	50	10.2	6.25	1.63
7	1st + Int +TD+ NM	$C(s) = 2.8\dfrac{(s^2 + 0.6239s + 0.0973)}{s}$	30	24.3	7.14	39	0.381	0.381	3.33
8	4th + TD	$C(s) = 0.9\dfrac{(s^2 + 0.7851s + 0.1541)}{s}$	30	8	23.7	43.2	0.394	0.394	2.82
9	2nd + UD	$C(s) = 340\dfrac{(s+20)^2}{s}$	10	0.13	Inf	81.4	260	260	1.1

The Table-4 presents the final results:

Table-4. Summary results of the designed PID controller for different OS% value specification.

TF	OS	Real OS	Ts	GM (sysK*sysG)	PM (dB) (sysK*sysG)	Wbw	Wc	Wbw/Wc
5	10	9.68	18.1	8.97	59.8	0.591	0.243	2.43
	20	22.4	21.6	6.51	51.2	0.679	0.332	2.05
	30	34.2	27.7	5.4	40.8	0.706	0.347	2.03
3	20	18.9	10.8	8.87	63.4	4.37	1.77	2.47
	30	29.8	8.85	8.14	57.3	4.84	2.36	2.05
	40	38.6	7.14	7.96	56.5	5.08	2.06	2.47
	50	49	3.75	10.8	43.1	3.23	1.52	2.13

Once again, an important conclusion is obtained with this experiment, because the relation between crossover and bandwidth frequencies was in average 2, despite the change in the OS% specification, confirming that the relationship between these frequencies is around 2.

Based on the results obtained, where the relation between overshoot and phase margin was compared for the analytical method (i.e. closed-loop response for a second order system) and simulation approach by means of FRtool the error was lower than 6%. This suggests that the analytical method developed here is a good approximation in order to obtain the phase margin as a function of the desired/designed overshoot.

In order to find a relationship between the overshoot and the phase margin, two practical experiments were performed: i) the first one using for each transfer function a PID controller designed for overshoot lower than 20%, and ii) having different PID controller for several values of overshoot in two transfer functions. The results of these experiments appear in Figure-8, where the overshoot and the phase margin are plotted. The trend presented for this data is close to that found in Figure-2. However, reading the exact values of overshoot in these results and getting the phase margin from Figure-2, it was found that for cases with time delay a higher error is present. This could be expected, since the approximation was done based in a second order system without time delay. Finally, it issuggested developing an analysis based

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in Nyquist criterion and the M circles, in order to get a validation of the relation of the controller design.
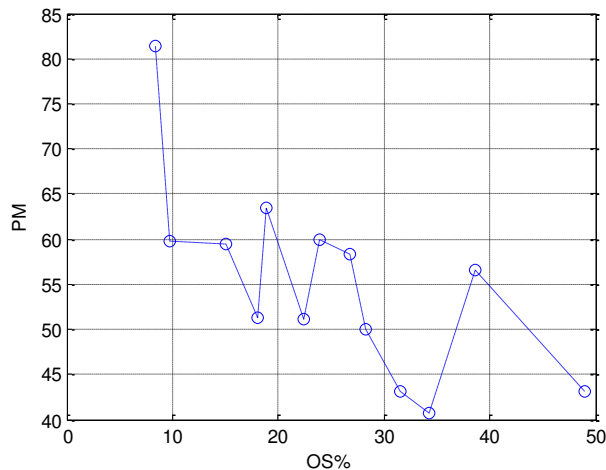

Figure-8. Relation obtained from practical experiments for OS% and PM.

Based on the experiment with PID control using FRtool for OS% as small as possible, the conclusion was that the relation between the bandwidth frequency (in closed-loop) and the crossover frequency (in open-loop) was 2.16.

By using the results for different overshoot values in two typical transfer functions, the relation was 2.23 for the same comparison.

Finally, it is proposed that the relation near to 2 for both experiments represents a good trade-off between analytical and practical methods.

REFERENCES

[1] K. J., Åström & T. Hägglund. 1995. PID Controllers: Theory, Design and Tuning. Instrument Society of America, Research Triangle Park, NC, USA.

[2] R. De Keyser & C. Ionescu. 2006. FRtool: a frequency response tool for CACSD in Matlab. In: Proceeding of the IEEE Conference on Computer Aided Control Systems Design, Munich, Germany. pp. 2275-2280.

[3] K.J., Åström & T. Hägglund. 1984. Automatic tuning of simple regulators with specifications on phase and amplitude margins, Automatica. 20, pp. 645-651.

[4] Ž. Juric & B. Perunicic. 2007. A method for closed-loop identification of Nyquist curve in three quadrants. Journal of Electrical engineering. 89(3): 251-261.

[5] P. Kumar & S. Majhi. 2009. Improved automatic tuning of PID controller for stable processes, ISA Transactions. 48(4): 423-427.

[6] I Naşcu, R De Keyser, S Folea, T Buzdugan. 2006. Development and Evaluation of a PID Auto-Tuning Controller. Proceeding of IEEE AQTR'06 International Conference on Automation, Quality and Testing, Robotics (Eds. L. Miclea, I. Stoian), ISBN 1-4244-0360-X, Cluj-Napoca, Romania. pp. 122-127.

[7] Ch. Anil & R. Padma Sree. 2005. Design of PID Controllers for FOPTD Systems with an Integrator and with/without a Zero. Indian Chem. Engr., Section A. 47(4): 235-242.

[8] Wang L and Cluett W. R. 1997. Tuning PID Controllers for Integrating Processes. IEE Proceeding. Control Theory and Applications. 144(5): 385-392.