# FOCAL AND GENERALIZED FOCAL SURFACES OF PARABOLIC CYLINDERS 

Georgi Hristov Georgiev ${ }^{1}$ and Milen Dimov Pavlov ${ }^{2}$<br>${ }^{1}$ Faculty of Mathematics and Informatics,Konstantin Preslavsky University, Universitetska Str.,Shumen, Bulgaria<br>${ }^{2}$ Department of Informatics and Mathematics, College Dobritch, Konstantin Preslavsky University,Dobritch, Bulgaria<br>E-mail:georgi.fmi@gmail.com


#### Abstract

Constructing a new surface from a given regular surface is an important task in Surface Modeling. A parallel surface (called also an offset surface) of a polynomial or rational surface is widely used in computer-aided design. The shape of a regular surface can be characterized by its principle curvatures, or equivalently, by its Gaussian and mean curvatures. The parabolic cylinder is a regular polynomial surface with many applications in product design. Parabolictrough solar collectors and parabolic cylindrical reflector antennas are good illustrations of the integrated form and function. In this paper, we present constructions of two cylindrical surfaces associated with an arbitrary parabolic cylinder. The first one is the unique focal surface of the parabolic cylinder which is defined by the reciprocal of the unique nonzero principle curvature. Its set of singular points coincides with the points of a straight line. The second one is a new regular surface obtained by the normals and reciprocal mean curvature of the original parabolic cylinder. This surface, called a generalized focal surface, consists of parabolic and planar points. A comparison of the properties of the parabolic cylinder, its focal surface, and its generalized focal surface is given. The relative positions of these three surfaces are also discussed.


Keywords: parabolic cylinder, principle curvatures, focal surface, mean curvature, generalized focal surface.

## 1. INTRODUCTION

One important task in surface modeling is obtaining a new surface from a given one. A parallel (or an offset) surface of a regular surface is the most popular case for such a construction. Any point of the parallel surface lies on the normal through a certain point of the original regular surface and the distance $d>0$ between these points is constant. The shape of a regular surface can be characterized by its principle curvatures, or equivalently, by its Gaussian and mean curvatures. There is another construction for a given regular surface with at least one nonzero principle curvature $\kappa$. The focal surface consists of points on the normals of the given surface such that the oriented distance between each point of the original surface and the corresponding point of the focal surface is equal to $\kappa^{-1}$. The last construction can be modified by replacing $\kappa^{-1}$ with some function of the two principle curvatures. In this case the new surface is called a generalized focal surface. Several studies have demonstrated that focal and generalized focal surfaces may possess important implications for manufacturing and computer-aided design. Radzevich [15] has investigated the focal surfaces, Hamman et al. [8] and Georgiev [2] examine the generalized focal surfaces.

The parabolic cylinder is a quadratic surface that has many applications in different branches of engineering and natural sciences. The shape of the parabolic cylinder is widely used in reflector antennas as pointed out by Schantzin [18], Song et al. [19], Stutzman and Thiele [20], and Yu et al. [23]. The parabolic cylinder plays a key role in the parabolic-trough solar collectors (see for instance [1], [10], [16], and [22]). There are other applications of the parabolic cylinder in physics as shown in [4] and [12]. The purpose of this paper is to investigate the properties of two cylindrical surfaces that are associated with any parabolic cylinder. The first surface is the unique focal
surface of the parabolic cylinder. The second one is the generalized focal surface of the parabolic cylinder which is obtained by the reciprocal mean curvature.

In the next section, main definitions and some facts concerning focal and generalized focal surfaces of a regular surface are presented. In addition, a specific polynomial parametrization of the parabolic cylinder is considered. After that the focal surface of the parabolic cylinder is analyzed. The reciprocal mean curvature-based generalized focal surface of the parabolic cylinder is also completely described. In the sequel, the properties of these two surfaces related to the parabolic cylinder are compared.

## 2. MATERIALS AND METHODS

### 2.1 Basic differential geometry of regular surfaces

Assume that $x, y$ and $z$ denote Cartesian coordinates of the points in the three dimensional Euclidean space $\mathbb{E}^{3}$. Let $S: D \rightarrow \mathbb{E}^{3}$ be be a regular surface of class $C^{2}$ defined on a domain $D \subseteq \mathbb{R}^{2}$ with a parametrization

$$
\begin{equation*}
\mathbf{r}(u, v)=(x(u, v), y(u, v), z(u, v))^{T}, \quad(u, v) \in D \tag{1}
\end{equation*}
$$

where (. $)^{T}$ denotes a transpose matrix. This means that the coordinate functions $x(u, v), y(u, v)$ and $z(u, v)$ have continuous partial derivatives up to order two. Moreover, at any point on the surface, the tangent vectors to the parametric curves $\mathbf{r}_{u}=\frac{\partial}{\partial u} \mathbf{r}(u, v)$ and $\mathbf{r}_{v}=\frac{\partial}{\partial v} \mathbf{r}(u, v)$ are non-collinear, or equivalently, the vector cross product $\mathbf{r}_{u} \times \mathbf{r}_{v}$ is a nonzero vector everywhere. Then, it is determined the unit normal vector field to the surface $S$
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$\mathbf{n}(u, v)=\frac{1}{\left\|\mathbf{r}_{u} \times \mathbf{r}_{v}\right\|}\left(\mathbf{r}_{u} \times \mathbf{r}_{v}\right)$,
where $(u, v) \in D$ and $\|$.$\| is a notation for a norm$ of a vector. The coefficients of the first fundamental form are expressed by three scalar (dot) products
$E=\mathbf{r}_{u} \cdot \mathbf{r}_{u}, \quad F=\mathbf{r}_{u} \cdot \mathbf{r}_{v}, \quad G=\mathbf{r}_{v} \cdot \mathbf{r}_{v}$.
At any point of the regular surface $S$ it is fulfilled $E G-F^{2}=\left\|\mathbf{r}_{u} \times \mathbf{r}_{v}\right\|^{2}>0$. From the existence of the second derivatives $\mathbf{r}_{u u}=\frac{\partial^{2}}{\partial u^{2}} \mathbf{r}(u, v), \mathbf{r}_{u v}=\frac{\partial^{2}}{\partial u \partial v} \mathbf{r}(u, v)$ and $\mathbf{r}_{v v}=\frac{\partial^{2}}{\partial v^{2}} \mathbf{r}(u, v)$, the coefficient of the second fundamental form
$L=\mathbf{n} \cdot \mathbf{r}_{u u}, \quad M=\mathbf{n} \cdot \mathbf{r}_{u v}, \quad N=\mathbf{n} \cdot \mathbf{r}_{v v}$
are also well defined. A parallel surface $S^{d}$ of $S$ at distance $d$ has a parametrization
$\mathbf{r}^{d}(u, v)=\mathbf{r}(u, v)+d . \mathbf{n}(u, v)$.
More details for this construction can be found in [5] and [17].

### 2.2 Focal surfaces of regular surfaces

Consider two $2 \times 2$ matrices formed from the coefficients of the first and second fundamental forms
$\mathbf{F}_{I}=\left(\begin{array}{ll}E & F \\ F & G\end{array}\right) \quad$ and $\quad \mathbf{F}_{I I}=\left(\begin{array}{ll}L & M \\ M & N\end{array}\right)$.
Then, the principle curvatures $\kappa_{1}$ and $\kappa_{2}$ of $S$ are the roots of the quadratic equation $\operatorname{det}\left(\mathbf{F}_{I I}-\kappa \mathbf{F}_{I}\right)=0$, or equivalently,
$\left(E G-F^{2}\right) \kappa^{2}-(E N+G L-2 F M) \kappa+\left(L N-M^{2}\right)=0$.
In [14] it has been shown that the discriminant of the above equation
$(E N+G L-2 F M)^{2}-4\left(E G-F^{2}\right)\left(L N-M^{2}\right)$
is greater than or equal to zero for any $(u, v) \in D$. Consequently, the two principle curvatures $\kappa_{1}$ and $\kappa_{2}$ are determined at any point of $S$. Recall that the Gaussian and mean curvature functions of the surface $S$ can be expressed by the coefficients of the first and second fundamental forms or by the principle curvatures as follows:
$K(u, v)=\frac{L N-M^{2}}{E G-F^{2}}=\kappa_{1} \kappa_{2}$
and
$H(u, v)=\frac{E N+G L-2 F M}{2\left(E G-F^{2}\right)}=\frac{1}{2}\left(\kappa_{1}+\kappa_{2}\right)$.
A point of a regular surface is called parabolic (resp. planar), if $K=0$ and $H \neq 0$ (resp. $K=0$ and $H=0$ ). For a surface $S$ with nonzero principle curvatures $\kappa_{1}$ and $\kappa_{2}$, there exist two surfaces $S^{e 1}$ and $S^{e 2}$ called evolutes or focal surfaces. They are given by
$\mathbf{r}^{e i}(u, v)=\mathbf{r}(u, v)+\frac{1}{\kappa_{i}} \mathbf{n}(u, v), \quad i=1,2$.
Properties of the focal surfaces are described in [5]. These surfaces have been developed for engineering applications by Radzevich [15]. Let us notice that any Euclidean motion of the Euclidean 3-space $\mathbb{E}^{3}$ is an affine transformation of $\mathbb{E}^{3}$ that preserves the scalar product and the orientation of $\mathbb{E}^{3}$. In particular, any Euclidean motion leaves invariant the coefficients of the first and second fundamental forms and the orientation of regular surfaces. Thus, the principle curvatures, the Gaussian and mean curvatures are also invariant under Euclidean motions. We can conclude that the image of a regular surface and the image of one of its focal surface under an arbitrary Euclidean motion form also a pair of a regular surface and one of its focal surfaces.

### 2.3 Generalized focal surfaces of regular surfaces

A generalization of the focal surfaces can be obtained by replacing in the equation (7) the reciprocal principle curvature with another real-valued function depending on $\kappa_{1}$ and $\kappa_{2}$. More precisely, this generalization, called a generalized focal surface $S^{g}$ to a regular surface $S$, is a parametric surface represented by
$\mathbf{r}^{g}(u, v)=\mathbf{r}(u, v)+g(u, v) \mathbf{n}(u, v)$,
where $\mathbf{r}(u, v)$ is a parametrization (1) of $S$, $\mathbf{n}(u, v)$ is the unit normal vector field (2) of $S$, and $g(u, v)$ is a real-valued function depending on the principal curvatures $\kappa_{1}$ and $\kappa_{2}$ of $S$. Such a kind of a surface has been introduced by Hagen and Hahmann in their paper [6]. An important type of a generalized focal surface (8) has been considered by the first author in [2]. See also [7] and [3]. Let $S$ be a regular surface of class $C^{2}$ with a parametrization (1). Suppose that $S$ has a nonzero mean curvature function $H$. Then, there exist a generalized focal surface $S^{g}$ given by
$\mathbf{r}^{g}(u, v)=\mathbf{r}(u, v)+\frac{1}{H} \mathbf{n}(u, v)$.
Other type of a generalized focal surface to a regular surface is introduced and studied by Moon in [13].

### 2.4 Direct similarities of the Euclidean 3-space

The Euclidean space $\mathbb{E}^{3}$ can be considered as an affine space with an associate vector space $\mathbb{R}^{3}$. This means
that the points of $\mathbb{E}^{3}$ can be identified with their position vectors.

Let us recall that the affine transformation $\Phi: \mathbb{E}^{3} \rightarrow \mathbb{E}^{3}$ is called a direct similarity, if for any point $\mathbf{x}=(x, y, z)^{T}$ in $\mathbb{E}^{3}$, its image $\Phi(\mathbf{x})$ in $\mathbb{E}^{3}$ is determined by the matrix equation
$\Phi(\mathbf{x})=\rho \mathbf{A x}+\mathbf{b}$,
where $\rho>0$ is a constant, $\mathbf{A}$ is a fixed $3 \times 3$ orthogonal matrix with $\operatorname{det}(\mathbf{A})=1$, and $\mathbf{b}=$ $\left(b_{1}, b_{2}, b_{3}\right)^{T} \in \mathbb{R}^{3}$ is a translation vector. The real positive number $\rho$ is called a similarity ratio. In other words, any direct similarity is an affine transformation of $\mathbb{E}^{3}$ into itself which preserves the orientation in $\mathbb{E}^{3}$ and the angels. If $\rho=1$, then the direct similarity (10) is a Euclidean motion. Consequently, the group $\operatorname{Sim}^{+}\left(\mathbb{E}^{3}\right)$ of all direct similarities of $\mathbb{E}^{3}$ is the minimal extension of the group $E M^{+}\left(\mathbb{E}^{3}\right)$ of the Euclidean motions of $\mathbb{E}^{3}$.

The generalized focal surfaces (9) are closely related to the direct similarities. Their main properties are proved in [2]:
i. If the generalized focal surfaces $S^{g}$ of $S$ is given by (9), then the set of all pairs $\left(S, S^{g}\right)$ is invariant under an arbitrary direct similarity $\Phi: \mathbb{E}^{3} \rightarrow \mathbb{E}^{3}$. This mean that if the surface $S_{\Phi}=\Phi(S)$ is given by a vector-valued function $\mathbf{r}_{\Phi}(u, v)$ and if the surface $S_{\Phi}^{g}=\Phi\left(S^{g}\right)$ is given by a vector-valued function $\mathbf{r}_{\Phi}^{g}(u, v)$, then
$\mathbf{r}_{\Phi}^{g}(u, v)=\mathbf{r}_{\Phi}(u, v)+\frac{1}{H_{\Phi}} \mathbf{n}_{\Phi}(u, v)$,
where $H_{\Phi}$ is a mean curvature of $S_{\Phi}$ and $\mathbf{n}_{\Phi}(u, v)$ is a unit normal vector field of $S_{\Phi}$.
ii. If the original surface $S$ is rational, then the generalized focal surface $S^{g}$ given by (9) is also rational.

### 2.5 A parametrization of the parabolic cylinder

The cylindrical surfaces form a subclass of the class of all ruled surfaces. The defining elements of an arbitrary ruled surface are its base curve and its rulings. The properties of the ruled surfaces are completely described in [5] and [14]. Up to a Euclidean motion, any parabolic cylinder can be considered as a ruled surface whose base curve is a parabola in the coordinate $x y$-plane and whose rulings are parallel to the $z$-axis. As in [18] and [19], we assume that the parabolic cylinder $S$ is the implicitly defined surface with the following equation in Cartesian coordinates
$x^{2}-4 f y=0$,
where $f>0$ is the focal length of the base parabola. Then, the straight line
$\left(\begin{array}{l}0 \\ f \\ t\end{array}\right), \quad t \in \mathbb{R}$
is called a focal line of $S$. Furthermore, the considered parabolic cylinder given by (11) possesses a parametrization
$\mathbf{r}(u, v)=\left(\begin{array}{c}2 f u \\ f u^{2} \\ v\end{array}\right)$,
$u \in(-\infty, \infty), \quad v \in(-\infty, \infty)$.
Now, we will recall some computations for this surface. The tangent vectors to the parametric curves
$\mathbf{r}_{u}(u, v)=(2 f, 2 f u, 0)^{T}$ and $\mathbf{r}_{v}(u, v)=(0,0,1)^{T}$
are linearly independent for any pair $(u, v) \in$ $\mathbb{R}^{2}$.In other words, the vectors $\mathbf{r}_{u}(u, v)$ and $\mathbf{r}_{v}(u, v)$ are non-collinear at any point of the surface parameterized by (12). Consequently, the parabolic cylinder is a regular surface of glass $C^{2}$. Every two parabolic cylinders $S_{1}$ and $S_{2}$ presented by (12) with focal distances $f_{1}$ and $f_{2}$, respectively, are similarity equivalent as point sets. In fact, the direct similarity (10) with $\rho=\frac{f_{2}}{f_{1}}, \mathbf{A}=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$ and $\mathbf{b}=(0,0,0)^{T}$ transforms $S_{1}$ into $S_{2}$.

## 3. RESULTS AND DISCUSSIONS

### 3.1 Properties of the focal surface of the parabolic cylinder

We continue with basic computations for the parabolic cylinder $S$ given by (12). Using (3) and (13), we see that the coefficients of the first fundamental form are
$E=4 f^{2}\left(1+u^{2}\right), \quad F=0, \quad G=1$.
The vector cross product $\mathbf{r}_{u}(u, v) \times \mathbf{r}_{v}(u, v)=$ $(2 f u,-2 f, 0)^{T}$ has a norm
$\left\|\mathbf{r}_{u}(u, v) \times \mathbf{r}_{v}(u, v)\right\|=2 f \sqrt{1+u^{2}}>0$.
Thus, the unit normal vector field (2) of the parabolic cylinder (12) is
$\mathbf{n}(u, v)=\left(\frac{u}{\sqrt{1+u^{2}}},-\frac{1}{\sqrt{1+u^{2}}}, 0\right)^{T}$.
From the second order partial derivatives $\mathbf{r}_{u u}(u, v)=(0,2 f, 0)^{T}, \mathbf{r}_{u v}(u, v)=(0,0,0)^{T}, \mathbf{r}_{v v}(u, v)=$ $(0,0,0)^{T}$, the coefficients (4) of the second fundamental form of $S$ regarding the parametrization (12) are
$L=-\frac{2 f}{\sqrt{1+u^{2}}}, \quad M=0, \quad N=0$.
Then, the quadratic equation (5) for the principle curvatures of $S$ becomes
$4 f^{2}\left(1+u^{2}\right) \kappa^{2}+\frac{2 f}{\sqrt{1+u^{2}}} \kappa=0$
Consequently, the principle curvatures of the parabolic cylinder (12) are $\kappa_{1}=\kappa_{\max }=0$ and
$\kappa_{2}=\kappa_{\min }=-\frac{1}{2 f\left(1+u^{2}\right) \sqrt{1+u^{2}}}$.
Thus, the parabolic cylinder has an exactly one focal surface (evolute). We can derive a parametrization of this focal surface.

Proposition 1.Let $S$ be the parabolic cylinder parameterized by (12), and let $S^{e}$ be its unique focal surface. Then, the surface $S^{e}$ is a cylindrical surface which is given by
$\mathbf{r}^{e}(u, v)=\left(\begin{array}{c}-2 f u^{3} \\ 3 f u^{2}+2 f \\ v\end{array}\right)$,
$u \in(-\infty, \infty), v \in(-\infty, \infty)$.
Proof. An arbitrary ruling of $S$ is parameterized by $\left(2 f u_{0}, f u_{0}^{2}, v\right)^{T}$, where $u_{0}$ is a real constant and $v \in(-\infty, \infty)$. Then, the principle curvatures (17) at the points of a fixed ruling of $S$ are one and the same. Similarly, the unit normal vectors (15) at the points of a fixed ruling of $S$ are also one and the same. Hence, there is a one parameter family of straight lines on the focal surface $S^{e}$ which is corresponding to the one parametric family of rulings of $S$. These straight lines are parallel to the $z$-axis, and therefore, $S^{e}$ is a cylindrical surface. Substituting the right-hand sides of (12), (15), and (17) into (7) yields $\mathbf{r}^{e}(u, v)=$
$\left(\begin{array}{l}2 f u \\ f u^{2} \\ v\end{array}\right)-2 f\left(1+u^{2}\right) \sqrt{1+u^{2}}\left(\begin{array}{c}\frac{u}{\sqrt{1+u^{2}}} \\ -\frac{1}{\sqrt{1+u^{2}}} \\ 0\end{array}\right)$.
After some calculations, we arrive at (18). $\square$
Now, we can analyse properties of the focal surface $S^{e}$. Since the tangent vectors to the parametric curves of $S^{e}$ are
$\mathbf{r}_{u}^{e}(u, v)=\left(-6 f u^{2}, 6 f u, 0\right)^{T}$,
$\mathbf{r}_{v}^{e}(u, v)=(0,0,1)^{T}$,
the focal surface $S^{e}$ is regular in the domain

$$
\begin{aligned}
D^{e} & =\{(-\infty, 0) \cup(0, \infty)\} \times(-\infty, \infty) \\
& =\{\mathbb{R} \backslash\{0\}\} \times \mathbb{R}
\end{aligned}
$$

Moreover, the unit normal vector field of $S^{e}$ is determined only on $D^{e}$ as follows
$\mathbf{n}^{e}(u, v)=\left(\frac{u}{\sqrt{u^{2}+u^{4}}}, \frac{u^{2}}{\sqrt{u^{2}+u^{4}}}, 0\right)^{T}, u \neq 0$.

By (19) the coefficient of the first fundamental form are
$E^{e}=36 f^{2} u^{2}\left(1+u^{2}\right), \quad F^{e}=0, \quad G^{e}=1$.
From the second order partial derivatives $\mathbf{r}_{u u}^{e}(u, v)=(0,2 f, 0)^{T}, \mathbf{r}_{u v}^{e}(u, v)=(0,0,0)^{T}, \mathbf{r}_{v v}^{e}(u, v)=$ $(0,0,0)^{T}$, we obtain the coefficients of the second fundamental form of $S^{e}$
$L^{e}=-\frac{6 f u^{2}}{\sqrt{u^{2}+u^{3}}}, \quad M^{e}=0, \quad N^{e}=0$.
Then, at regular points of $S^{e}$, the Gaussian curvature $K^{e}$ is equal to zero, and the mean curvature
$H^{e}=-\frac{u^{2}}{12 f\left(u^{2}+u^{3}\right) \sqrt{u^{2}+u^{3}}}$
takes only negative values. Consequently, the focal surface $S^{e}$ on the domain $D^{e}$ contains only parabolic points. The points on the straight line $\lambda(v)=\mathbf{r}^{e}(0, v)=$ $(0,2 f, v)^{T}, \quad v \in \mathbb{R}$ are the singular points of $S^{e}$ because both vectors $\mathbf{r}_{u}^{e}(0, v)$ and $\mathbf{r}_{u}^{e}(0, v) \times \mathbf{r}_{v}^{e}(0, v)$ are equal to $(0,0,0)^{T}$.

The base curve of $S^{e}$ in the $x y$-plane is the semi-cubical parabola
$\beta^{e}(u)=\left(-2 f u^{3}, 3 f u^{2}+2 f, 0\right)^{T}, u \in(-\infty, \infty)$
which is implicitly defined by
$f\left(\frac{x}{2}\right)^{2}=\left(\frac{y-2 f}{3}\right)^{3}$.
This curve is the evolute of the base parabola
$\beta^{o}(u)=\left(2 f u, f u^{2}, 0\right)^{T}, u \in(-\infty, \infty)$.
of the original parabolic cylinder $S$. Obviously, the implicit polynomial equation (22), considered in the Euclidean 3-space, defines the focal surface $S^{e}$. According to [11], the same surface $S^{e}$ is called a right cylindrical surface with a base curve (directrix) semi-cubical parabola, and the straight line $\lambda(v)$ of its singular points is called a cuspidal edge of $S^{e}$. Such a kind of a surface is used in recent investigations in singularity theory as [9] and [21].

We can also determine the points of the intersection $S \cap S^{e}$. Since both surfaces are cylinders whose rulings are parallel to the $z$-axis, it is sufficient to find the intersection points of their base curves $\beta^{o}(u)$ and $\beta^{e}(u)$ in the $x y$-plane. For any point $P \in \beta^{o}(u) \cap \beta^{e}(u)$ there are parametric values $u^{o}, u^{e} \in \mathbf{R}$ such that $\beta^{o}\left(u^{o}\right)=$ $\beta^{e}\left(u^{e}\right)$. By (21) and (23) these values constitute a real solution of the nonlinear system
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$$
\left\lvert\, \begin{align*}
& 2 f u^{o}=2 f\left(u^{e}\right)^{3} \\
& f\left(u^{o}\right)^{2}=f\left(u^{e}\right)^{2}+2 f \tag{24}
\end{align*}\right.
$$

The above system can be reduced to the equation $\left(u^{e}\right)^{6}-\left(u^{e}\right)^{2}-2=0$, which is equivalent to
$\left(\left(u^{e}\right)^{2}+1\right)\left(\left(u^{e}\right)^{4}-\left(u^{e}\right)^{2}-2\right)=0$.
The last equation has two real roots $u_{1}^{e}=\sqrt{2}$ and $u_{2}^{e}=-\sqrt{2}$. It immediately follows that the pairs $\left(u_{1}^{o}=\right.$ $\left.-2 \sqrt{2}, u_{1}^{e}=\sqrt{2}\right)$ and $\left(u_{2}^{o}=2 \sqrt{2}, u_{2}^{e}=-\sqrt{2}\right)$ are the real solutions of the system (24). In other words, the intersection $\beta^{o}(u) \cap \beta^{e}(u)$ of the base curves consists of two points $P_{1}$ and $P_{2}$ whose position vectors are $(-4 \sqrt{2} f, 8 f, 0)^{T}$ and $(4 \sqrt{2} f, 8 f .0)^{T}$, respectively. Then, the intersection $S \cap S^{e}$ of the original parabolic cylinder with its focal surface falls into two straight lines parallel to $z$-axis which are parameterized by $(-4 \sqrt{2} f, 8 f, v)^{T}, v \in \mathbb{R}$ and $(4 \sqrt{2} f, 8 f, v)^{T}, v \in \mathbb{R}$. In addition, $S^{e}$ meets $S$ transversally because the init normal vectors $\mathbf{n}^{e}(u, v)$ given by (20) ) and $\mathbf{n}(u, v)$ given by (15) ) are noncollinear at any point of the intersection $S \cap S^{e}$. A part of the original parabolic cylinder with $f=3$, the corresponding part of the focal surface, and their intersection are plotted in Figure-1.


Figure-1.The intersection(two segments in white) of the part of the parabolic cylinder (in red) and irs foca surface (in blue).

### 3.2 A generalized focal surface of the parabolic cylinder

Let us consider again the parabolic cylinder $S$ defined by (12). Its Gaussian curvature $K=\kappa_{1} \kappa_{2}$ is identically zero because of $\kappa_{1}=0$. Substituting (14) and (16) into (6) we get the expression
$H=-\frac{1}{4 f \sqrt{1+u^{2}}\left(1+u^{2}\right)}$
for the mean curvature of $S$. Thus, the function $H(u, v)=$ $\frac{\kappa_{2}}{2}$ never vanishes and takes only negative values. First, we
can find the generalized focal surface of $S$ defined by the reciprocal mean curvature of $S$.

Theorem 2.Let $S$ be the parabolic cylinder parameterized by (12), and let $S^{g}$ be its generalized focal surface obtained by (9). Then, the surface $S^{g}$ possesses a polynomial parametrization
$\mathbf{r}^{g}(u, v)=\left(\begin{array}{c}-2 f u-4 f u^{3} \\ 4 f+5 f u^{2} \\ v\end{array}\right)$,
where $(u, v) \in\{(-\infty, \infty) \times(-\infty, \infty)\}=\mathbb{R}^{2}$. Moreover, $S^{g}$ is a cylindrical surface which is regular everywhere on $\mathbb{R}^{2}$.
Proof. From (15) and (25) it follows that

$$
\begin{aligned}
\frac{1}{H} \mathbf{n}(u, v) & =-4 f\left(1+u^{2}\right) \sqrt{1+u^{2}}\left(\begin{array}{c}
\frac{u}{\sqrt{1+u^{2}}} \\
-\frac{1}{\sqrt{1+u^{2}}} \\
0
\end{array}\right) \\
& =\left(\begin{array}{c}
-4 f\left(1+u^{2}\right) u \\
4 f\left(1+u^{2}\right) \\
0
\end{array}\right) .
\end{aligned}
$$

Then, the right-hand side of (9) can be written as
$\left(\begin{array}{c}2 f u \\ f u^{2} \\ v\end{array}\right)+\left(\begin{array}{c}-4 f\left(1+u^{2}\right) u \\ 4 f\left(1+u^{2}\right) \\ 0\end{array}\right)$
$=\left(\begin{array}{c}-2 f u-4 f u^{3} \\ 4 f+5 f u^{2} \\ v\end{array}\right)$.
This proves (26). If $u_{0}$ is a real constant, then any parametric curve
$\left(-2 f u_{0}-4 f u_{0}^{3}, 4 f+5 f u_{0}^{2}, v\right)^{T}, \quad v \in \mathbb{R}$
of $S^{g}$ is a straight line parallel to the $z$-axis. Therefore, $S^{g}$ is a cylindrical surface. The tangent vectors to the parametric curves of $S^{g}$
$\mathbf{r}_{u}^{g}(u, v)=\left(-2 f-12 f u^{2}, 10 f u, 0\right)^{T}$
$\mathbf{r}_{v}^{g}(u, v)=(0,0,1)^{T}$
are non-collinear for any pair $(u, v) \in \mathbb{R}^{2}$. In fact, for both $u=0$ and $u \neq 0$, the vector cross product
$\mathbf{r}_{u}^{g}(u, v) \times \mathbf{r}_{v}^{g}(u, v)=\left(10 f u, 2 f+12 f u^{2}, 0\right)^{T}$
is differen from $(0,0,0)^{T}$. Consequently, $S^{g}$ is a regular surface.

Second, we can examine some specific properties of the generalized focal surface $S^{g}$ with a parametrization (26). Using (27), we find scalar and vector quantities of
$S^{g}$. Suppose that $A(u)=(5 u)^{2}+\left(1+6 u^{2}\right)^{2}>0$. Then, the coefficients of the first fundamental form are
$E^{g}=4 f^{2} A(u), F^{g}=0, G^{g}=1$,
the unit normal vector field is
$\mathbf{n}^{g}(u, v)=\left(\frac{5 u}{\sqrt{A(u)}}, \frac{1+6 u^{2}}{\sqrt{A(u)}}, 0\right)^{T}$,
and the second order derivatives are $\mathbf{r}_{u u}^{g}(u, v)=$ $(-24 f u, 10 f, 0)^{T}, \quad \mathbf{r}_{u v}^{g}(u, v)=(0,0,0)^{T}, \quad \mathbf{r}_{v v}^{g}(u, v)=$ $(0,0,0)^{T}$. Furthermore, at the points of $S^{g}$, the coefficients of the second fundamental form are
$L^{g}=-\frac{10 f\left(1-6 u^{2}\right)}{\sqrt{A(u)}}, \quad M^{g}=0, \quad N^{g}=0$,
the Gaussian curvature $K^{g}$ is identically zero, and by (6) the mean curvature $H^{g}$ is
$H^{g}(u, v)=-\frac{10 f\left(1-6 u^{2}\right)}{4 f\left[(5 u)^{2}+\left(1+6 u^{2}\right)^{2}\right]^{3 / 2}}$.
Hence, the generalized focal surface $S^{g}$ consists of parabolic and planar points. The parabolic points of $S^{g}$ correspond to parametric values $u \in \mathbb{R}\left\{\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right\}$, because at these points both conditions $K^{g}=0$ and $H^{g} \neq 0$ are fulfilled. If $u_{1}=\frac{1}{\sqrt{6}}$ and $u_{2}=-\frac{1}{\sqrt{6}}$, then $H^{g}\left(u_{1}, v\right)=0$ and $H^{g}\left(u_{2}, v\right)=0$ for any $v \in \mathbb{R}$. Thus, all planar points of $S^{g}$ parameterized by (26) are the points on two rulings of $S^{g}$ determined by
$\left(-\frac{2 f}{\sqrt{6}}-\frac{2 f}{3 \sqrt{6}}, 4 f+\frac{5 f}{6}, v\right)^{T}=\left(-\frac{8 f}{3 \sqrt{6}}, \frac{29 f}{6}, v\right)^{T}, \quad v \in \mathbb{R}$ $\operatorname{and}\left(\frac{2 f}{\sqrt{6}}+\frac{2 f}{3 \sqrt{6}}, 4 f+\frac{5 f}{6}, v\right)^{T}=\left(\frac{8 f}{3 \sqrt{6}}, \frac{29 f}{6}, v\right)^{T}, v \in \mathbb{R}$.

Now, we will describe the intersection of the original parabolic cylinder $S$ given by (12) and its generalized focal surface $S^{g}$ defined by (26). For any point $Q \in S \cap S^{g}$ there are pairs $\left(u^{o}, v^{o}\right),\left(u^{g}, v^{g}\right) \in \mathbb{R}^{2}$ such that $\mathbf{r}\left(u^{o}, v^{o}\right)=\mathbf{r}^{g}\left(u^{g}, v^{g}\right)$, or equivalently,

$$
\left\lvert\, \begin{aligned}
& 2 f u^{o}=-2 f u^{g}-4 f\left(u^{g}\right)^{3} \\
& f\left(u^{o}\right)^{2}=4 f+5 f\left(u^{g}\right)^{2} \\
& v^{o}=v^{g} .
\end{aligned}\right.
$$

The last system can be rewritten as

$$
\left\lvert\, \begin{align*}
& u^{o}=-u^{g}-2\left(u^{g}\right)^{3} \\
& \left(u^{o}\right)^{2}=4+5\left(u^{g}\right)^{2}  \tag{29}\\
& v^{o}=v^{g}
\end{align*}\right.
$$

Eliminating $u^{0}$ from the first two equations we get a sixth degree equation
$\left(u^{g}\right)^{2}+4\left(u^{g}\right)^{4}+4\left(u^{g}\right)^{6}=4+5\left(u^{g}\right)^{2}$
in the unknown $u^{g}$. This equation is equivalent to the equation

$$
\left[\left(u^{g}\right)^{2}-1\right]\left[\left(u^{g}\right)^{2}+1\right]^{2}=0
$$

with real roots $u_{1}^{g}=1$ and $u_{2}^{g}=-1$. Substituting these values into the first equation of (29) yields $u_{1}^{o}=-3$ and $u_{2}^{o}=3$. Consequently, the intersection $S \cap S^{g}$ consists of two straight lines $(-6 f, 9 f, v)^{T}, v \in \mathbb{R}$ and $(6 f, 9 f, v)^{T}, v \in \mathbb{R}$ which are two common rulings of $S$ and $S^{g}$. By (15) and (28) the normal vectors $\mathbf{n}(u, v)$ of $S$ and $\mathbf{n}^{g}(u, v)$ of $S^{g}$ are noncollinear at any point of these common rulings. Therefore, $S^{g}$ meets $S$ transversally. In case of $f=3$, the intersection of a part of the parabolic cyliner wirh the corresponding generalized focal surface and the planar points on this generalized focal surface are plotted in Figure-2.


Figure-2. The intersection (two segments in white) of a part of the parabolic cylinder (in red) with its generalized focal surface (in green) and the planar points (two segments in yellow ) on this generalized focal surface.

Finally, we derive the implicit equation of $S^{g}$. The first two coordinate functions in the parametrization (26) can be written as
$\left\lvert\, \begin{aligned} & x=x(u, v)=-2 f u-4 f u^{3} \\ & y=y(u, v)=4 f+5 f u^{2} \geq 4 f\end{aligned}\right.$.
Then, we have
$\left\lvert\, \begin{aligned} & x^{2}=4 f^{2} u^{2}\left(1+u^{2}\right)^{2} \\ & \frac{y-4 f}{5 f}=u^{2} \geq 0\end{aligned}\right.$.
Eliminating $u^{2}$ from these two equalities gives

$$
\left(\frac{x}{2 f}\right)^{2}=\frac{(y-4 f)(2 y-3 f)^{2}}{(5 f)^{3}} \quad(y \geq 4 f)
$$

The equation above is an implicit representation of $S^{g}$.

### 3.3 Relative position between the focal surface $S^{e}$ and the generalized focal surfaceS ${ }^{\mathbf{g}}$

According to (11), the interior (resp. exterior) of the parabolic cylinder $S$ consists of the points in $\mathbb{E}^{3}$ whose Cartesian coordinates $(x, y, z)$ satisfy the inequality $x^{2}-4 f y<0$ (resp. $x^{2}-4 f y>0$ ). Then, the points of $S^{e}$ that belong to the interior of $S$ are determined by (18) with $-\sqrt{2}<u<\sqrt{2}$. Similarly, the points of $S^{g}$ that belong to the interior of $S$ are determined by (26) with $-1<u<1$. Using (22) we say that the interior of the focal surface $S^{e}$ consists of the points in $\mathbb{E}^{3}$ whose Cartesian coordinates ( $x, y, z$ ) satisfy the inequality
$f\left(\frac{x}{2}\right)^{2}-\left(\frac{y-2 f}{3}\right)^{3}<0$.
From (26) the Cartesian coordinates of the points on $S^{g}$ are $x=-2 f u-4 f u^{3}, y=4 f+5 f u^{2}$, and $z=v$. Then for the points on $S^{g}$, the left-hand side of inequality (30) can be written as
$f\left(\frac{-2 f u-4 f u^{3}}{2}\right)^{2}-\left(\frac{4 f+5 f u^{2}-2 f}{3}\right)^{3}$
$=\left(\frac{f}{3}\right)^{3}\left[27 u^{2}\left(1+2 u^{2}\right)^{2}-\left(2+5 u^{2}\right)^{3}\right]$
$=\left(\frac{f}{3}\right)^{3}\left(-8-33 u^{2}-42 u^{4}-17 u^{6}\right)<0$.
Hence, the surfaces $S^{e}$ and $S^{g}$ have no common points, and all points of $S^{g}$ belong to the interior of $S^{e}$. The mutual position between $S^{e}$ and $S^{g}$ is illustrated in Figure-3.


Figure-3. The generalized focal surface (in green) is placed inside the focal surface (in blue).

## 4. CONCLUSIONS

In this paper, we describe two cylindrical surfaces associated to an arbitrary parabolic cylinder $S$. The first
one is the unique focal surface $S^{e}$ of $S$. All regular points of $S^{e}$ are parabolic, and all singular points of $S^{e}$ lie on one of its rulings. The second one is the generalized focal surface $S^{g}$ which is defined by the normals and reciprocal mean curvature of the original parabolic cylinder $S$. All points of $S^{g}$ are regular and can be divided into two types. The planar points of $S^{g}$ coincide with the points on two rulings of $S^{g}$. All other points on $S^{g}$ are parabolic. Starting with a suitable polynomial parametrization of $S$, we obtain polynomial parameterizations of $S^{e}$ and $S^{g}$. The relative positions of $S, S^{e}$ and $S^{g}$ are discussed. In particular, it is shown that all points of $S^{g}$ belong to the interior of $S^{e}$.

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## REFERENCES

[1] L.S. Conrado, A. Rodriguez-Pulido and G. Calderon. 2017. Thermal performance of parabolic trough solar collectors. Renewable and Sustainable Energy Reviews. 67:1345-1359.
[2] G. H. Georgiev. 2012. Rational generalized offsets of rational surfaces. Mathematical Problems in Engineering. volume 2012, Article ID 618148: 1-15.
[3] G.H.Georgiev and M.D. Pavlov. 2017. Curvature Dependent Offsets to Elliptic Cones. Far East Journal of Mathematical Sciences. 102(11): 2757-2783.
[4] N. Graham, A. Shpunt, T. Emig, S. J. Rahi, R. L. Jaffe and M. Kardar. 2011. Electromagnetic Casimir forces of parabolic cylinder and knife-edge geometries Physical Review D. 83(12): 12500701-12500714.
[5] A. Gray, E. Abbena and S. Salamon. 2006. Modern Differential Geometry of Curves and Surfaces with Mathematica, Third edition. Studies in Advanced Mathematics. Chapman and Hall/CRC, Boca Raton, FL.
[6] H. Hagen and S. Hahmann. 1992. Generalized focal surfaces: A new method for surface interrogation. Proceedings Visualization '92. IEEE Computer Society Press. Boston MA,USA. pp. 70-76, doi:10.1109/VISUAL.1992.235224.
[7] H. Hagen, S. Hahmann and T. Schreiber. 1995.Visualization and Computation of Curvature Behaviour of Free-Form Curves and Surfaces, Computer-Aided Design. 27(7): 545-552.
[8] S. Hahmann,A. Belyaev, L. Buse, G. Elber, B. Mourrain and C. Roessl. 2008. Shape interrogation. In: L. De Floriani, M. Spagnuolo (Eds.), Shape Analysis and Structuring. Springer Berlin, pp. 1-51.
[9] S. Izumiya, K. Saji and N. Takeichi. 2017. Flat surfaces along cuspidal edges. Jourbal of Singularities. 16: 73-100.
[10] V.K. Jebasingh and G.M. Joselin-Herbert. 2016. A review of solar parabolic trough collector. Renewable and Sustainable Energy Reviews. 54: 1085-1091.
[11] S.N. Krivoshapko and V. N. Ivanov. 2015. Encyclopedia of Analytical Surfaces. Springer, Heidelberg.
[12] K. Kurihara, A. Otomo, K. Yamamoto, J. Takahara, M. Tani, and F. Kuwashima. 2015. Identification of Plasmonic Modes in Parabolic Cylinder Geometry by Quasi-Separation of Variables, Plasmonics. 10: 165182.
[13]H. P. Moon. 2009. Equivolumetric offset surfaces, Computer Aided Geometric Design. 26(1): 17-36.
[14]N. M. Patrikalakis and T. Maekawa. 2002. Shape Interrogation for Computer Aided Design and Manufacturing. Springer-Verlag, Heidelberg.
[15] S. P. Radzevich. 2014. Generation of Surfaces: Kinematic Geometry of Surface Machining. CRC Press/Taylor \& Francis Group, Boca Raton.
[16]G.A. Salazar , N. Fraidenraich, C.A. Alves de Oliveira, O. Castro Vilela, M. Hongn, and J. Gordon. 2017. Analytic modeling of parabolic trough solar thermal power plants. Energy. 138(1):1148-1156.
[17] A. Saxena and B. Sahay. 2005. Computer Aided Engineering Design. Springer/Anamaya Publishers, New Delhi.
[18]H.G. Schantz. 2015. The Art and Science of Ultrawideband Antennas, Second Edition. Artech House, Boston.
[19] K. Song, J. Kim, S. Hur, J.-H. Kwak, S.-H. Lee and T. Kim. 2016. Directional Reflective Surface Formed via Gradient-Impeding Acoustic Meta-Surfaces. Scientific Reports. volume 6, Article number:32300, 1-9.
[20] W. L. Stutzman and G. A. Thiele. 2012. Antenna Theory and Design, $3^{\text {rd }}$ Edition, John Wiley and Sons, USA.
[21]K. Teramoto. 2016. Parallel and dual surfaces of cuspidal edges. Differential Geometry and its Applications. 44:52-62.
[22]L. Xu, Z.F. Wang, G.F. Yuan, F.H. Sun, and X.L. Zhang. 2015. Thermal performance of parabolic trough solar collectors under the condition of dramatically varying DNI. Energy Procedia. 69: 218 225.
[23]T. Yu, F-1. Guan and L. Dai. 2014. Design and analysis of bidirectional deployable parabolic cylindrical antenna. Journal of Zhejiang University Science A: Applied Physics and Engineering. 15(2): 83-96.

