



# A COMPARISON OF MLE METHOD AND OLSE FOR THE ESTIMATION OF MODIFIED WEIBULL DISTRIBUTION PARAMETERS BY USING THE SIMULATION

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## ABSTRACT

In this paper, we study the Maximum Likelihood Estimation (MLE) and Ordinary Least Squares Estimator (OLSE) methods for estimation of the unknown parameters of the modified Weibull distribution. A comparison between these estimators is provided by using extensive simulation and mean squared error criteria to select the best method. Computational experiments on the presented methods are reported.

**Keywords:** modified Weibull distribution, maximum likelihood estimator, least squares method.

## INTRODUCTION

The probability density function of any random variable  $t$  having a modified Weibull distribution (MWD) with scale parameter  $\alpha > 0$  and both shape parameters  $\beta \geq 0$  and  $\lambda > 0$  is given by

$$f(t; \alpha, \beta, \lambda) = \begin{cases} \alpha(\beta + \lambda t)^{\beta-1} \exp(-\alpha t - \alpha t^\beta e^{\lambda t}) & t \geq 0 \\ 0 & \text{o.w.} \end{cases} \quad (1)$$

If  $\beta = 0$ , the resulting distribution is called the type 1 extreme-value which is also known as a log-gamma distribution when  $\lambda = 0$  then MWD reduces to the two-parameter Weibull distribution. Also when both  $\beta = 0$  and  $\lambda = 0$  then MWD reduces to one-parameter Rayleigh distribution. The modified Weibull model was developed by Xie *et al.* (2003) [1] this lifetime distribution is an important feature for reliability analysis.

Vasile *et al.* [3] studied the method of Bayes to estimate the parameters of the MWD and Upadhyaya and

$$L(\alpha, \beta, \lambda; t_1, t_2, \dots, t_n) = \alpha^n \prod_{i=1}^n (\beta + \lambda t_i)^{\beta-1} \exp \sum_{i=1}^n (\lambda t_i - \alpha t_i^\beta e^{\lambda t_i}) \quad (4)$$

and the logarithm for the likelihood function

$$\ln L = n \ln \alpha + \sum_{i=1}^n \ln(\beta + \lambda t_i) + (\beta - 1) \sum_{i=1}^n \ln t_i + \sum_{i=1}^n (\lambda t_i - \alpha t_i^\beta e^{\lambda t_i}) \quad (5)$$

The partial derivatives for the log-likelihood function with respect to unknown parameters  $\alpha, \beta$  and  $\lambda$ , and placing them to zero are following the equations

$$\frac{n}{\hat{\alpha}} - \sum_{i=1}^n t_i^\beta e^{\hat{\lambda} t_i} = 0 \quad (6)$$

Gupta [2] using Markov chain Monte Carlo simulation to studied the Bayes analysis of the MWD. Ateya [4] study the estimation problem of the censored sample of order statistics generalized from MWD.

The cumulative distribution function and reliability function respectively are

$$F(t; \alpha, \beta, \lambda) = 1 - \exp(-\alpha t^\beta e^{\lambda t}) \quad (2)$$

$$R(t) = \exp(-\alpha t^\beta e^{\lambda t}) \quad (3)$$

We review two methods in the objective analytical procedure; the maximum likelihood estimation (MLE) and the ordinary least squares method (OLSM). These methods are compared in Section 4, using the mean square error (MSE) criteria.

### Maximum Likelihood Estimator (MLE)

The likelihood function for Modified Weibull distribution is.

$$\sum_{i=1}^n \frac{1}{(\hat{\beta} + \hat{\lambda} t_i)} + \sum_{i=1}^n \ln t_i - \hat{\alpha} \sum_{i=1}^n (t_i^\beta e^{\hat{\lambda} t_i} \ln t_i) = 0 \quad (7)$$

$$\sum_{i=1}^n \frac{t_i}{(\hat{\beta} + \hat{\lambda} t_i)} + \sum_{i=1}^n t_i - \hat{\alpha} \sum_{i=1}^n (t_i^{\hat{\beta}+1} e^{\hat{\lambda} t_i}) = 0 \quad (8)$$



Observing that the three equations (6),(7) and (8) are difficult to solve, then it is impossible to find MLE for  $\alpha$ ,  $\beta$  and  $\lambda$  directly, we use the

$$J_k = \begin{bmatrix} \frac{-n}{\alpha^2} & -\sum_{i=1}^n t_i^\beta e^{\lambda t_i} \ln t_i & -\sum_{i=1}^n t_i^{\beta+1} e^{\lambda t_i} \\ -\sum_{i=1}^n t_i^\beta e^{\lambda t_i} \ln t_i & -\sum_{i=1}^n \frac{1}{(\beta + \lambda t_i)^2} - \alpha \sum_{i=1}^n t_i^\beta e^{\lambda t_i} (\ln t_i)^2 & -\sum_{i=1}^n \frac{t_i}{(\beta + \lambda t_i)^2} - \alpha \sum_{i=1}^n t_i^{\beta+1} e^{\lambda t_i} \ln t_i \\ -\sum_{i=1}^n t_i^{\beta+1} e^{\lambda t_i} & -\sum_{i=1}^n \frac{t_i}{(\beta + \lambda t_i)^2} - \alpha \sum_{i=1}^n t_i^{\beta+1} e^{\lambda t_i} \ln t_i & -\sum_{i=1}^n \frac{t_i^2}{(\beta + \lambda t_i)^2} - \alpha \sum_{i=1}^n t_i^{\beta+2} e^{\lambda t_i} \end{bmatrix}$$

Which must be a non-singular symmetric matrix so its inverse can be found.

$$\begin{bmatrix} \alpha_{k+1} \\ \beta_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} \alpha_k \\ \beta_k \\ \lambda_k \end{bmatrix} - J_k^{-1} \cdot \begin{bmatrix} \frac{n}{\hat{\alpha}} - \sum_{i=1}^n t_i^{\hat{\beta}} e^{\hat{\lambda} t_i} \\ \sum_{i=1}^n \frac{1}{(\hat{\beta} + \hat{\lambda} t_i)} + \sum_{i=1}^n \ln t_i - \hat{\alpha} \sum_{i=1}^n (t_i^{\hat{\beta}} e^{\hat{\lambda} t_i} \ln t_i) \\ \sum_{i=1}^n \frac{t_i}{(\hat{\beta} + \hat{\lambda} t_i)} + \sum_{i=1}^n t_i - \hat{\alpha} \sum_{i=1}^n (t_i^{\hat{\beta}+1} e^{\hat{\lambda} t_i}) \end{bmatrix} \quad (9)$$

The absolute value for the difference between the new values for  $\alpha$ ,  $\beta$  and  $\lambda$  in new iterative value with previous value for  $\alpha$ ,  $\beta$  and  $\lambda$  in last iterative represent the error term, its symbol is  $\varepsilon$ , which is a very small and assumed value .and  $\alpha_0$ ,  $\beta_0$  and  $\lambda_0$  are the initial values which are assumed.

### Ordinary Least Squares Estimator (OLSE)

The ordinary least squares method is one of themost popular procedures for estimating the parameters when the model is linear or non-linear in variables. Every linear or non-linear model involve response variable denoted by ( $Y$ ) which is effected by explanatory variable denoted by ( $x$ ).

If the relationship between variable ( $Y$ ) and variable ( $x$ ) is linear , the model is represented mathematically by straight line equation as follows:

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon \quad (10)$$

Where:  $\beta_0$  represents the intercept term,  $\beta_1$  represents the slop term of  $x_1$ ,  $\beta_2$  represents the slop term of  $x_2$  and  $\varepsilon$  represents the error term.

The idea of this method is to minimize the sum of squared differences between observed sample values and the estimate expected values by linear approximation:

$$\varepsilon_i = y_i - \beta_0 - \beta_1 x_1 - \beta_2 x_2 \quad (11)$$

numericalprocedure is Newton-Raphson method, with the following Jacobian matrix.

$$\sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n [y_i - \hat{y}_i]^2 \quad (12)$$

$$\sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n [y_i - \hat{\beta}_0 - \hat{\beta}_1 x_1 - \hat{\beta}_2 x_2]^2 \quad (13)$$

By using the CDF of modified Weibull distribution (2) which are as follows:

$$1 - [F(t_i)] = \exp(-\alpha t_i^\beta e^{\lambda t_i}) \quad (14)$$

By taking the double logarithm of above equation getting

$$\ln(-\ln[1 - \{F(t_i)\}]) = \ln \alpha + \beta \ln t_i + \lambda t_i \quad (15)$$

Comparing the above equation with the simple linear model  $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$  we get:

$$Y = \ln(-\ln[1 - \{F(t_i)\}]), x_1 = \ln t_i, x_2 = t_i,$$

$$\beta_0 = \ln \alpha, \beta_1 = \beta, \beta_2 = \lambda \text{ and}$$

$$\varepsilon = \ln(-\ln[1 - \{F(t_i)\}]) - \ln \alpha - \beta \ln t_i - \lambda t_i \quad (16)$$

Where  $F(t_i)$  is empirical cumulative distribution function, then we can find it by using the following formula [5]?



$$F(t_i) = \frac{i - 0.5}{n}$$

By taking the sum square of above equation for the two sides to reach:

$$\sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n [\ln(-\ln[1 - \{F(t_i)\}]) - \ln \alpha - \beta \ln t_i - \lambda t_i]^2 \quad (17)$$

Then:

$$\frac{\partial(\sum_{i=1}^n \varepsilon_i^2)}{\partial \alpha} = \frac{-2}{\alpha} \sum_{i=1}^n \ln(-\ln[1 - \{F(t_i)\}]) + \frac{2n}{\alpha} \ln \alpha + \frac{2\beta}{\alpha} \sum_{i=1}^n \ln t_i + \frac{2\lambda}{\alpha} \sum_{i=1}^n t_i \quad (18)$$

$$\sum_{i=1}^n \ln(-\ln[1 - \{F(t_i)\}]) (\ln t_i) - \ln \hat{\alpha}_{LS} \sum_{i=1}^n \ln t_i - \hat{\beta}_{LS} \sum_{i=1}^n (\ln t_i)^2 - \hat{\lambda}_{LS} \sum_{i=1}^n t_i \ln t_i = 0 \quad (22)$$

$$\sum_{i=1}^n \ln(-\ln[1 - \{F(t_i)\}]) (t_i) - \ln \hat{\alpha}_{LS} \sum_{i=1}^n t_i - \hat{\beta}_{LS} \sum_{i=1}^n t_i \ln t_i - \hat{\lambda}_{LS} \sum_{i=1}^n t_i^2 = 0 \quad (23)$$

Let

$$S_1 = \sum_{i=1}^n \ln(-\ln[1 - \{F(t_i)\}]), S_2 = \sum_{i=1}^n t_i, S_3 = \sum_{i=1}^n \ln(t_i)$$

$$S_4 = \sum_{i=1}^n \ln(-\ln[1 - \{F(t_i)\}]) (\ln t_i), S_5 = \sum_{i=1}^n (\ln t_i)^2,$$

$$S_6 = \sum_{i=1}^n t_i \ln(t_i), S_7 = \sum_{i=1}^n \ln(-\ln[1 - \{F(t_i)\}]) (t_i) \text{ and}$$

$$S_8 = \sum_{i=1}^n t_i^2.$$

the equations (21), (22) and (23) are a system of nonlinear equations and can be solved simultaneously with respect to  $\hat{\alpha}_{LS}$ ,  $\hat{\beta}_{LS}$  and  $\hat{\lambda}_{LS}$  by:

$$\hat{\alpha}_{LS} = \exp\left(\frac{S_1 S_5 S_8 - S_1 S_6^2 + S_2 S_4 S_6 - S_2 S_5 S_7 - S_3 S_4 S_8 + S_3 S_6 S_7}{n S_5 S_8 - n S_6^2 - S_2^2 S_5 + 2 S_2 S_3 S_6 - S_3^2 S_8}\right) \quad (24)$$

$$\hat{\beta}_{LS} = \frac{n S_4 S_8 - n S_6 S_7 + S_1 S_2 S_6 - S_1 S_3 S_8 - S_2^2 S_4 + S_2 S_3 S_7}{n S_5 S_8 - n S_6^2 - S_2^2 S_5 + 2 S_2 S_3 S_6 - S_3^2 S_8} \quad (25)$$

$$\frac{\partial(\sum_{i=1}^n \varepsilon_i^2)}{\partial \beta} = -2 \sum_{i=1}^n \ln(-\ln[1 - \{F(t_i)\}]) (\ln t_i) + 2 \ln \alpha \sum_{i=1}^n \ln t_i + 2\beta \sum_{i=1}^n (\ln t_i)^2 + 2\lambda \sum_{i=1}^n t_i \ln t_i \quad (19)$$

$$\frac{\partial(\sum_{i=1}^n \varepsilon_i^2)}{\partial \lambda} = -2 \sum_{i=1}^n \ln(-\ln[1 - \{F(t_i)\}]) (t_i) + 2 \ln \alpha \sum_{i=1}^n t_i + 2\beta \sum_{i=1}^n t_i \ln t_i + 2\lambda \sum_{i=1}^n t_i^2 \quad (20)$$

We place the partial derivatives (18), (19) and (20) to zero n as simpler we get

$$\sum_{i=1}^n \ln(-\ln[1 - \{F(t_i)\}]) - n \ln \hat{\alpha}_{LS} - \hat{\beta}_{LS} \sum_{i=1}^n \ln t_i - \hat{\lambda}_{LS} \sum_{i=1}^n t_i = 0 \quad (21)$$

$$\hat{\lambda}_{LS} = \frac{-n S_4 S_6 + n S_5 S_7 - S_1 S_2 S_5 + S_1 S_3 S_6 + S_2 S_3 S_4 - S_3^2 S_7}{n S_5 S_8 - n S_6^2 - S_2^2 S_5 + 2 S_2 S_3 S_6 - S_3^2 S_8} \quad (26)$$

### Monte Carlo Simulation

This section presents the Monte Carlo simulation to comparison with MLE and OLSE methods for the parameters of modified Weibull distributions. The MSE for parameters are calculated using 43,800 simulated samples.

All computations in this simulation are performed using MATLAB R2014a. We consider sample sizes n = 10, 25 and 50, while scale parameters are taken  $\alpha=0.1$ , 0.05 for the modified Weibull distributions. Also, the first shape parameter is taken  $\beta=0.1$ , 0.2 and the second shape parameter is taken  $\lambda=0.1$ .

We are taken initial values  $\alpha=1$ ,  $\beta=1$  and  $\lambda=1$  for MLE with an error less than 0.0001 as a numerical Newton-Raphson method in this study. And by using the inversion of cumulative distribution function we can generate random numbers from the modified Weibull distribution as follows:

$$F(t; \alpha, \beta, \lambda) = 1 - \exp(-\alpha t^\beta e^{\lambda t}) = u$$



$$t = \exp\left(-\frac{\lambda \cdot \exp\left(\frac{\ln\left(-\frac{\ln(1-u)}{\beta}\right)}{\beta}\right)}{\beta} \beta - \ln\left(\ln\left(-\frac{\ln(1-u)}{\beta}\right)\right)\right) \quad (27)$$

Where  $u \sim N(0, 1)$  and  $LambertW$  is a function that satisfies the following relation:

In this study we assumed two model, the first model is  $(\alpha = 0.1, \beta = 0.1, \lambda = 0.1)$  and the second

model is  $(\alpha = 0.05, \beta = 0.2, \lambda = 0.1)$ . And we replicate the data of experiment N times with sample size n, the results of simulation presented in the following Tables.

**Table-1.** MSE of scale parameter  $\alpha$  for the first model.

Cr. Meth. n	MSE		Best	N
	MLE	OLSE		
10	0.810000207389752	0.008885448140488	OLSE	200
	0.810000207677127	0.013866373901629	OLSE	500
	0.810000193072404	0.599808416179078	OLSE	1000
	0.810000187443670	0.074946334157825	OLSE	2000
	0.810000191648963	1.434487755805187	MLE	2100
	0.810000205085839	33.306061793229844	MLE	2500
25	0.810000197197269	0.002622131278485	OLSE	200
	0.810000204608555	0.003054352139546	OLSE	500
	0.810000202717642	0.002870944321338	OLSE	1000
	0.810000210626075	0.003256681733394	OLSE	2000
	0.810000209322569	0.003107117135591	OLSE	2100
	0.810000202010869	0.002767917602914	OLSE	2500
50	0.810000236024901	0.001364779069256	OLSE	200
	0.810000232436911	0.001135551871666	OLSE	500
	0.810000238294715	0.001324533716744	OLSE	1000
	0.810000238961582	0.001373229563785	OLSE	2000
	0.810000242616599	0.001375085314148	OLSE	2100
	0.810000233383592	0.001332754979112	OLSE	2500



**Table-2.** MSE of shape parameter  $\beta$  for the first model.

Cr.	MSE		Best	N
	MLE	OLSE		
10	0.210682878367625	2.452850147380615	MLE	200
	0.212295569584633	1.581722867938437	MLE	500
	0.211720628187839	3.687800519305009	MLE	1000
	0.212726727479227	2.576017409094774	MLE	2000
	0.211886212834559	3.853256469799561	MLE	2100
	0.212520075058526	2.747400979217147	MLE	2500
25	0.233912859975677	0.195394038449581	OLSE	200
	0.233962424078517	0.121966148592804	OLSE	500
	0.233499542771420	0.099272037734120	OLSE	1000
	0.233384325835870	0.193670846375020	OLSE	2000
	0.233497899779575	0.100731384671951	OLSE	2100
	0.233648738525361	0.107462206628891	OLSE	2500
50	0.242914520250060	0.009715745206370	OLSE	200
	0.243147741496383	0.009281542930936	OLSE	500
	0.242749650317884	0.005770903983325	OLSE	1000
	0.243173315890615	0.013309766551080	OLSE	2000
	0.242848425890481	0.010716144876984	OLSE	2100
	0.243213902639503	0.014337832831609	OLSE	2500

**Table-3.** MSE of shape parameter  $\lambda$  for the first model.

Cr.	MSE		Best	N
	MLE	OLSE		
10	0.840962439813026	0.006301387919544	OLSE	200
	0.840470396805303	0.004448715233655	OLSE	500
	0.840624076071195	0.007954038229538	OLSE	1000
	0.840357592954846	0.006816527519235	OLSE	2000
	0.840555671695671	0.008250480921981	OLSE	2100
	0.840435585143504	0.006901981581571	OLSE	2200
25	0.835221277321123	0.000661016873576	OLSE	200
	0.835231341353647	0.000611682569224	OLSE	500
	0.835345972879849	0.000558246431627	OLSE	1000
	0.835372171223390	0.000797735891366	OLSE	2000
	0.835345150335956	0.000589354009478	OLSE	2100
	0.835305911771138	0.000590121377076	OLSE	2500
50	0.833329999417076	0.000210139059744	OLSE	200
	0.833289378169277	0.000222878604907	OLSE	500
	0.833364670571798	0.000200377157629	OLSE	1000
	0.833278698848937	0.000220466543243	OLSE	2000
	0.833350384737919	0.000229489214562	OLSE	2100
	0.833266623205731	0.000230091628830		2500

**Table-4.** MSE of scale parameter  $\alpha$  for the second model.

Cr.	MSE		Best	N
	MLE	OLSE		
10	0.902500098401846	0.029228744152917	OLSE	200
	0.902500095984256	0.050069144259683	OLSE	500
	0.902500106372802	67527.56726731632	MSE	1000
	0.902500110732227	1342.363075338750	MSE	2000
	0.902500115616728	12969.10284445154	MSE	2100
	0.902500109685206	52410968885.07371	MSE	2500
25	0.902500122613167	0.001092350605799	OLSE	200
	0.902500117389027	0.001441840422322	OLSE	500
	0.902500128126531	0.001290867819185	OLSE	1000
	0.902500128924866	0.001233891126218	OLSE	2000
	0.902500131638134	0.001365435294614	OLSE	2100
	0.902500131109104	0.001189096393779	OLSE	2500
50	0.902500145811476	0.000658377293165	OLSE	200
	0.902500154692685	0.000552235356731	OLSE	500
	0.902500154774002	0.000592530620799	OLSE	1000
	0.902500160374250	0.000569793942161	OLSE	2000
	0.902500156643863	0.000615476438409	OLSE	2100
	0.902500155102598	0.000529849670358	OLSE	2500

**Table-5.** MSE of shape parameter  $\beta$  for the second model.

Cr.	MSE		Best	N
	MLE	OLSE		
10	0.148368537186429	14.870217457267277	MLE	200
	0.147322894926476	4.983387461536311	MLE	500
	0.146710817186559	8.701473897550000	MLE	1000
	0.146890382745876	7.754145005462000	MLE	2000
	0.147184208173416	5.060997062760000	MLE	2100
	0.147136187937897	6.944540000000000	MLE	2500
25	0.161358308782946	1.051557176488829	MLE	200
	0.161272181185863	0.610027403960937	MLE	500
	0.160509096113500	0.462059250261181	MLE	1000
	0.160612134579963	0.504859524830287	MLE	2000
	0.160908318861960	0.633789402345871	MLE	2100
	0.160373700767689	0.521674940403557	MLE	2500
50	0.167666057747652	0.108334953688489	OLSE	200
	0.167223420795141	0.097159299246970	OLSE	500
	0.167172019754857	0.116922128733864	OLSE	1000
	0.166808284286324	0.093298218044138	OLSE	2000
	0.167164106481752	0.112413706103683	OLSE	2100
	0.167191887006069	0.068275003287531	OLSE	2500

**Table-6.** MSE of shape parameter  $\lambda$  for the second model.

Cr.	MSE		Best	N
	MLE	OLSE		
10	0.835053263898479	0.025177153575600	OLSE	200
	0.835332026065178	0.010348708017170	OLSE	500
	0.835502204042962	0.014151906340000	OLSE	1000
	0.835449800031967	0.013692182253000	OLSE	2000
	0.835409497843725	0.010110020070000	OLSE	2100
	0.835365230947213	0.012900000000000	OLSE	2500
25	0.831624040924874	0.002301474053352	OLSE	200
	0.831641911075780	0.001501589636618	OLSE	500
	0.831827605579875	0.001266515629803	OLSE	1000
	0.831808235082367	0.001263079459064	OLSE	2000
	0.831738667604273	0.001496231567379	OLSE	2100
	0.831862955533916	0.001353367258287	OLSE	2500
50	0.830139959873007	0.000435696235886	OLSE	200
	0.830236618821057	0.000347556186021	OLSE	500
	0.830239745231603	0.000415870255460	OLSE	1000
	0.830325000766765	0.000369533583806	OLSE	2000
	0.830246037738241	0.000419328763536	OLSE	2100
	0.830237283913493	0.000339365921982	OLSE	2500

Note that we can make the following comments for the results in the above tables:

a) The results of simulation presented in Table-1 is following conclusions may be summarized: the OLSR is best method except when  $n=10$  with  $N=2100, 2500$  according to the MSE criterion of  $\hat{\alpha}$ .

b) The results of simulation presented in Table 2 is following conclusions may be summarized: the MLE is best method when  $n=10$  with  $N=200, 500, 1000, 2100, 2500$  and except that thy OLSE is best, according to the MSE criterion of  $\hat{\beta}$ .

c) The results of simulation presented in Table-3 is following conclusions may be summarized: the only OLSR is best method according to the MSE criterion of  $\hat{\lambda}$ .

d) The results of simulation presented in Table-4 is following conclusions may be summarized: the OLSR is best method except when  $n=10$  with  $N=1000, 2000, 2100, 2500$  according to the MSE criterion of  $\hat{\alpha}$ .

e) The results of simulation presented in Table-5 is following conclusions may be summarized: the MLE is best method except when  $n=50$  with  $N=1000, 2000, 2100, 2500$  according to the MSE criterion of  $\hat{\beta}$ .

f) The results of simulation presented in Table-6 is following conclusions may be summarized: the only OLSR is best method according to the MSE criterion of  $\hat{\lambda}$ .

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