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BENDING MODELING OF T-BEAMS WITH STRAIGHT HAUNCHES SUBJECTED TO DISTRIBUTED UNIFORM LOAD USING MAXIMA

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ABSTRACT

In this study, we implement a matrix calculation system in Maxima (GNU) based on the numerical integration of flexibility terms to obtain fixed-end rotation, fixed-end moments and the entries of the T-beams stiffness matrix with linear haunches. We took into account the shear and axial deformations that are not included in formulations of packages for broadly used structural calculation on a global scale as SAP2000. All of this, to demonstrate that a systematic and reasonable standard-setting calculation methodology based on numerical integration can match the results obtained from robust finite element tools, being this methodology a more economical option because of its low computational cost. For this matter, the numerical integrals of the flexibility and rotation factors of Tena-Colunga were calculated approximately, implementing three-point Gauss quadrature's, Taylor polynomial and Romberg, with the help of the Computational Algebra System Maxima(GNU) for tapered forms ofT-beams". We established a comparison with the formulations used by the finite element packages Ansys® and SAP2000® and we validated the results with identical models made in these two packages. Efficacy and precision of the developed model were verified by using this approach, and we concluded that this methodology represents significant advantages in the structural calculation of tapered beams with linear haunches.

Keywords: non-prismatic, T-beam, maxima (GNU), numerical integration, SAP2000®, Ansys®, flexibility integrals.

1. INTRODUCTION

One of the main purposes of structural engineering during the last decades has been to propose reliable elastic analysis methods that allow to satisfactorily modeling the elements of variable section, so that they have clear span on the determination of internal forces, deformations and displacements that allow to adequately designing this type of elements. One of the main problems in the analysis of structures with variable moment of inertia along its length is to find the end moments and their stiffness factors.

In the 1950s and 1960s, design aids and design tables such as the well-known PCA (Portland Cement Association-Handbook, 1958) were developed, which allowed obtaining stiffness constants and fixed-end moments of variable section elements. At that time, due to the limitations of calculating computational effort, the PCA tables used hypotheses to simplify the problem; among the most important, were, to consider the variation of the haunches stiffness (linear, parabolic or cubic, based on the type of haunch geometry) depending on the main moment of inertia in flexion, considering it independent of the cross section. However, further studies on the subject showed that these premises were not true. The assumptions of the PCA also disregarded the shear deformations and the span - height relation of the beam in the definition of the various stiffness factors, and these simplifications generated in some cases significant errors in the determination of stiffness factors (Tena-Colunga, 1996). Tena-Colunga defined the two-dimensional and three-dimensional stiffness elastic matrices of variable section elements based on the classical Bernoulli - Euler beam theory and the flexibility method, introducing the axial and shear deformations, as well as the shape of the cross section and, based on that, proposed new design aids for tapered beams.

The method of Tena-Colunga uses a clear formulation that allows its almost direct application and with the feasibility of programming a reasonable method since the definition of beam elements of two-dimensional and three-dimensional variable sections is manageable using the flexibility method proposed by the author.

The complexity of calculating the stiffness matrix of variable section elements using matrix methods due to the need of implementing numerical integration belongs to the past, because today structural calculation is reasonably manageable due to the large development that has had the computing field on the software and hardware level. The fact that the analyst can implement his own numerical integrals using the help provided by tools of free-use computational algebra as Maxima(GNU) and explicitly formulate the matrix calculation of a structural system of tapered beams by the stiffness method makes the study of this topic very interesting.

This article presents the results of a research work where the proposal formulated by Tena-Colunga was adopted to model tapered beams using the Bernoulli - Euler theory as the basic principle. This methodology for modeling tapered beams allows achieving very reliable results when compared with finite element formulations. This paper compared the results of the analysis of tapered T-beams implementing the matrix displacement method having to resort to the calculation of the stiffness matrix factors by calculating the numerical integration of the flexibility matrix elements in the free software Maxima(GNU). It was demonstrated that with the calculation possibilities that the processors offer us today and with the advances in mathematical calculation tools, it is possible to solve problems of relevance in structural engineering without the help of a highly expensive commercial finite element software. The proposed mathematical model was validated with the programs SAP2000 ® and Ansys ® to establish the effectiveness of the implemented method.

2. METHODS

2.1 Stiffness matrix in local coordinates of the non-prismatic element

The stiffness matrix in local coordinates of the non-prismatic element is defined using the flexibility method, implementing a calculation system in the Computational Algebra software Maxima (GNU) that allows solving the numerical integration problem of the entries of a matrix. The basic flexibility matrix of variable section elements is presented in equation (1).

$$[f] = \begin{Bmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & f_{23} \\ 0 & f_{32} & f_{33} \end{Bmatrix} \tag{1}$$

Where equations (2), (3), (4), (5), and (6) describe the entries of the flexibility matrix.

$$f_{11} = \int_0^l \frac{dz}{EA(z)} \tag{2}$$

$$f_{22} = \int_0^l \frac{z^2 dz}{EI_x(z)} + \int_0^l \frac{dz}{GA_{cy}(z)} \tag{3}$$

$$f_{23} = \int_0^l \frac{zdz}{EI_X(z)} = f_{32} \tag{4}$$

$$f_{33} = \int_0^l \frac{dz}{EI_X(z)} \tag{5}$$

For its part, the stiffness matrix is obtained inverting the flexibility submatrix, so their entriesare implicitly defined. The global stiffness matrix in local coordinates of the two-node beam-column element in Figure-1 is expressed according to equation (6).

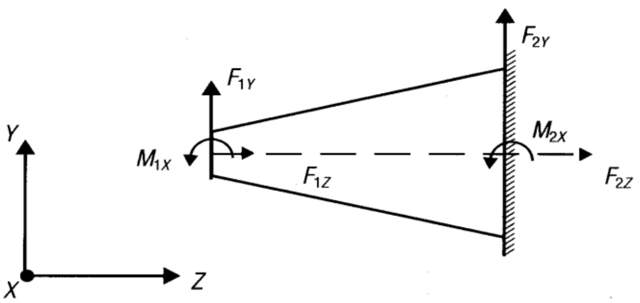

Figure-1. Variable-section two-dimensional beam element. Source: [6]

$$[K] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{6}$$

Equations (7), (8), (9), and (10) determine stiffness sub-arrays.

$$[k_{11}] = \begin{Bmatrix} r_{az} & 0 & 0 \\ 0 & r_{aax} & r_{abx} \\ 0 & r_{abx} & r_{11x} \end{Bmatrix} \tag{7}$$

$$[k_{12}] = \begin{Bmatrix} -r_{az} & 0 & 0 \\ 0 & -r_{aax} & r_{bax} \\ 0 & -r_{abx} & r_{12x} \end{Bmatrix} \tag{8}$$

$[k_{21}] = [k_{12}]^T$(9)

$$[k_{22}] = \begin{Bmatrix} r_{az} & 0 & 0 \\ 0 & r_{aax} & -r_{bax} \\ 0 & -r_{bax} & r_{22x} \end{Bmatrix} \tag{10}$$

The seven (7) different entries of these submatrices are defined by equations (11) to (18).

$$r_{az} = \frac{1}{f_{11}} \tag{11}$$

$$Det_x = f_{22}f_{33} - f_{33^2} \tag{12}$$

$$r_{11x} = \frac{f_{22}}{Det_X} \tag{13}$$

$$r_{12x} = \frac{f_{23}L - f_{22}}{Det_X} \tag{14}$$

$$r_{22x} = \frac{f_{33}L^2 - 2f_{23}L + f_{22}}{Det_X} \tag{15}$$

$$r_{aax} = \frac{r_{11x} + r_{22x} + 2r_{12x}}{L^2} \tag{16}$$

$$r_{abx} = \frac{r_{11x} + r_{12x}}{L} \tag{17}$$

$$r_{bax} = \frac{r_{22x} + r_{12x}}{L} \tag{18}$$

2.2Matrix system of stiffness equations in local coordinates

The matrix system of equations to be resolved in local coordinates is represented by the equation (19).

$$\begin{Bmatrix} [k_{11}][k_{l2}] \\ [k_{21}][k_{22}] \end{Bmatrix} \begin{Bmatrix} \{u_1\} \\ \{u_2\} \end{Bmatrix} = \begin{Bmatrix} \{F_1\} \\ \{F_2\} \end{Bmatrix} \tag{19}$$

Where the force vectors {F} and displacement {u} of the matrix system (19) are defined by equations (20) to (23).

$$\{u_1\} = \begin{Bmatrix} u_{1z} \\ u_{1y} \\ \theta_{lx} \end{Bmatrix} \tag{20}$$

$$\{F_1\} = \begin{Bmatrix} F_{1z} \\ F_{1y} \\ M_{1x} \end{Bmatrix} \tag{21}$$

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$$\{u_2\} = \begin{Bmatrix} u_{2z} \\ u_{2y} \\ \theta_{2x} \end{Bmatrix} \quad (22)$$

$$\{F_2\} = \begin{Bmatrix} F_{2z} \\ F_{2y} \\ M_{2x} \end{Bmatrix} \quad (23)$$

Once the stiffness matrix of the variable section element is defined in local coordinates, it is implanted in a calculation system (matrix or finite element). The element stiffness matrix in global coordinates is obtained using transformation matrices, and the inter-element connectivity is defined by the assembly rule.

2.3 Fixed-end rotation and end moments of the non-prismatic element under a uniform distributed load

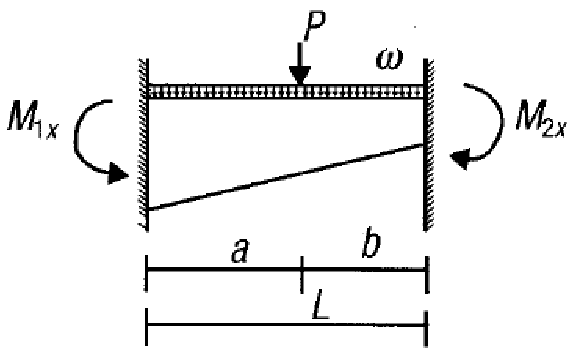

Figure-2. Recessed variable section beam subjected to general load. Source [6]

In a double-recessed variable-section beam subjected to a general load condition in its main flexion plane as in Figure-2, the fixed-end rotation can be determined by means of the method of the conjugated beam in Figure-3 applying equilibrium and taking into account the shear deformations. The rotations at the beam-ends 1 and 2 are calculated according to the equations (24) and (25).

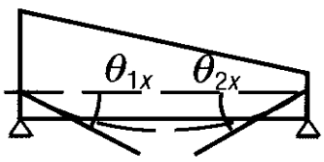

Figure-3. Conjugated beam. Source [6]

$$\theta_{2x} = \frac{1}{L}\int_0^L \frac{zM_{0x}}{EI_X(z)}dz + \frac{1}{L}\int_0^L \frac{V_{0y}}{GA_{cy}(z)}dz \quad (24)$$

$$\theta_{1x} = \int_0^L \frac{M_{0x}}{EI_X(z)}dz - \theta_{2x} \quad (25)$$

And the end moments in the main direction of flexion are calculated, according to Figure-4, by means of the equations (26) and (27).

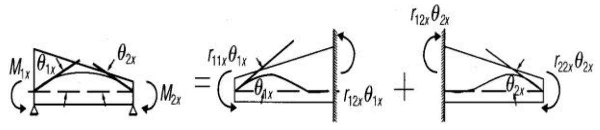

Figure-4. Fixed end moments in functionof rotations and rigidities of the beam. Source [6].

$$M_{1x} = r_{11x}\theta_{1x} - r_{12x}\theta_{2x} \quad (26)$$

$$M_{2x} = r_{22x}\theta_{2x} - r_{12x}\theta_{1x} \quad (27)$$

For a uniformly distributed load in the main flexion plane (Ω x), it is known that the moment and shear equations of the corresponding isostatic structure are those shown in equations (28) and (29).

$$M_{0x} = \frac{\omega_x L}{2}z - \frac{\omega_x z^2}{2} \quad (28)$$

$$V_{0y} = \frac{\omega_x L}{2} - \omega_x z \quad (29)$$

Therefore, the fixed end rotations are shown in equations (30) and (31).

$$\theta_{2x} = \frac{\omega_x}{2E}\left[\int_0^L \frac{z^2dz}{I_x(z)} - \frac{1}{L}\int_0^L \frac{z^3dz}{I_x(z)}\right] + \frac{\omega_x}{G}\left[\frac{1}{2}\int_0^L \frac{dz}{A_{cy}(z)} - \frac{1}{L}\int_0^L \frac{zdz}{A_{cy}(z)}\right] \quad (30)$$

$$\theta_{1x} = \frac{\omega_x}{2E}\left[L\int_0^L \frac{zdz}{I_x(z)} - \int_0^L \frac{z^2dz}{I_x(z)}\right] - \theta_{2x} \quad (31)$$

By replacing the fixed-end rotations (30) and (31) in equations (26) and (27), the fixed end moments of the variable section beam are obtained.

2.4 Mathematical model of T-beam with straights haunches

In Figure-5 a T-beam is shown in elevation and its cross section with constant roller width b_f, roller thickness t_f and beam web thickness b_w. The beam web height h_w varies in a linear way in three different sections.

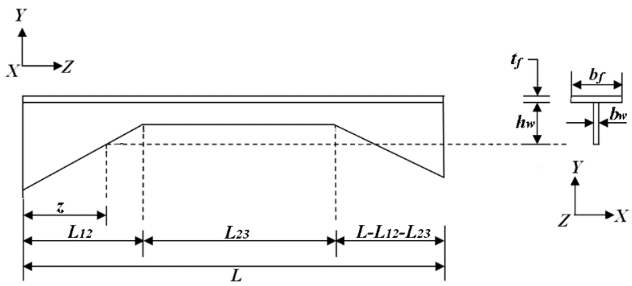

Figure-5. Cross section T with straight haunches.

The constant dimensions of the section in meters are b_f =1.10, t_f=0.05, b_w=0.3, h_{w1}= h_{w4}=0.65, h_{w2}= h_{w3}=0.40;and the material properties are E=1581139 T/m^{2}and G=E/2.4. The lengths of the tapered sections in meters are L_{12}=2.35 and L_{23}=2.10; and the total length of the beam is L=7.125.The uniform distributed load in its entire length is ω_x=0.5 Ton/m.

The linear variation of the beam web height h_w in the first and last haunch, is mathematically represented in equations (32) and (33).

First interval, for$0 \leq z \leq L_{12}$

$$h_w(z) = \left(\frac{h_{w2}-h_{w1}}{L_{12}}\right) z + h_{w1} \tag{32}$$

Third interval, for$L_0 \leq z \leq L$, where $L_0 = L_{12} + L_{23}$

$$h_w(z) = \left(\frac{h_{w4}-h_{w3}}{L_{34}}\right)(z - L_0) + h_{w3} \tag{33}$$

The area properties *A(z)*, the effective shear area $A_{cy}(z)$,the centroid distance *Y(z)* and the moment of inertia of the cross section $I_x(z)$, are presented as functions of (z) in equations (34) to (37).

$$A(z) = b_w h_w(z) + b_f t_f \tag{34}$$

$$A_{cy}(z) = b_w(h_w(z) + t_f) \tag{35}$$

$$Y(z) = \frac{\left(b_f \frac{t_f^2}{2}\right) + [b_w h_w(z)]\left[\left(\frac{h_w(z)}{2}\right) + t_f\right]}{A(z)} \tag{36}$$

$$I_x(z) = \frac{[b_f t_f{}^3 + b_w h_w(z)^3]}{3} - A(z)\left[Y(z) - t_f\right]^2 \tag{37}$$

The implementation of the calculation system in Maxima (GNU) is summarized in the following stages:

- Data input:h_{w1}, h_{w2}, h_{w3}, h_{w4}, b_w, b_f, t_f, *L*, L_{12}, L_{23}, *E*, *G*.
- Definition of variables for each section: $h_w(z)$, *A(z)*, $A_{cy}(z)$, *Y(z)*, *I(z)*.
- Calculation of flexibility terms integrals and factors of rotation for each section: Approximation by Taylor polynomials, Approximation by Gaussian quadrature of three points, Approximation of Romberg.
- Summation of the flexibility terms of the tapered beam.
- Summation of the rotation terms of the tapered beam.
- Calculation of the fixed-end rotation for uniform distributed load.
- Calculation of the entries of the stiffness matrix depending on the flexibility factors.
- Calculation of fixed-end moments for uniform distributed load ω_x.
- Calculation of the shear forces at the ends of the beam $V_{1y}$and V_{2y}.
- Tapered beam stiffness matrix assembly.
- Fixed end forces vector assembly.
- Stiffness matrix equation: [Fn-FFn]=[Knn][δn].
- Determination of displacement [δn].
- Stiffness matrix equation: reactions: [Fa]=[Kan][δn].

2.4 Computational modeling inSAP2000®and Ansys®

Modeling the problem of T-beam with straight haunches in the program Ansys ®is done with the finite element BEAM188. This element is based on the theory of Timoshenko beams and includes shear deformation effects. BEAM188 is a two-node or quadratic three-dimensional linear beam element that has six degrees of freedom on each node. It is suitable for linear applications and can be used with any beam cross-section even with haunching. The finite element model of the tapered T-beam made in Ansys®is shown in Figure-6.

Unlike Ansys® an analysis in SAP2000 ® only considers bending and axial deformations, while shear deformations are ignored. This situation causes the results of the rotations at the ends of the beam to vary slightly as shown in the Results Table-2.

In SAP2000® the tapered T-beam model is made with a two-node "Frame" type bar element that includes the haunching effect of the cross-section. The distributed load is applied to the beam length and under a static analysis the degrees of freedom and the reactions at the end of the nodes and the internal element forces are obtained.

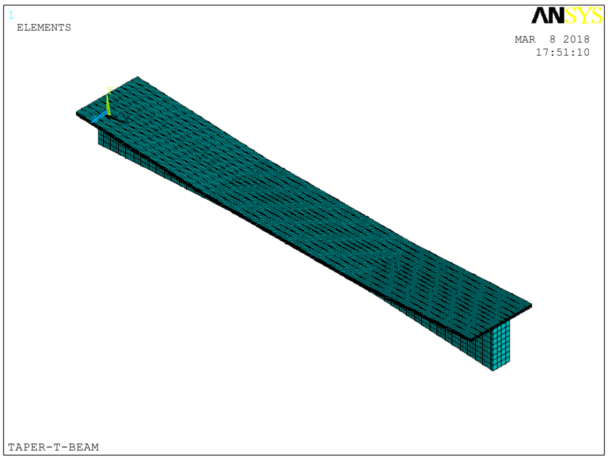

Figure-6. Tapered T-beam model in Ansys®.

The finite element model of the tapered T-beam made in SAP2000®is shown in Figure-7.

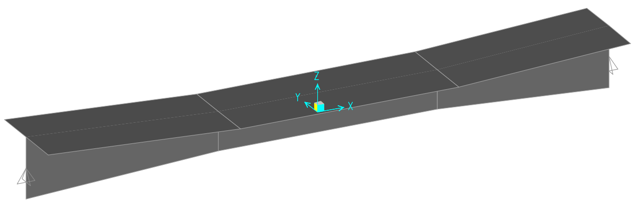

Figure-7. Tapered T-beam model inSAP2000®.

The variation of the non-prismatic element section in SAP2000®is shown in Figure-8. The equation (38) calculates the variation of inertia I(z) used by SAP2000®and represents the haunch on the beam along its length. In equation (38) *n* represents the type of variation of the haunches, being n=1linear variation, n=2 parabolic variation and n=3 cubic variations.

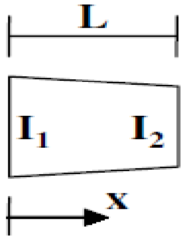

Figure-8. Inertia variation inSAP2000®.

$$I(x) = \left[\left(I_1^{(1/n)}\right)\left(1 - \frac{x}{L}\right) + \left(I_2^{(1/n)}\right)\left(\frac{x}{L}\right)\right] \quad (38)$$

3.RESULTS AND DISCUSSIONS

Numerical integration was implemented in a calculation program specially developed in the free CAS Maxima (GNU) to solve the problem of a T beam ofwith three haunches subjected to distributed uniform load (ω_x)along its length. The results of the mathematical model implemented in the Maxima (GNU), are compared with identical models of finite elements in the programs Ansys® y SAP2000®, finding an important coincidence that leads us to conclude that the mathematical calculus of flexibility matrices integration is an alternative and simple method that is presented as an economical and efficient option for the analysis of tapered elements. Three-point Gauss quadrature integration techniques, Romberg and Taylor polynomial were used for calculating flexibility and rotation factors.

In Table-1, the results of the flexibility factors for the first haunch $0 \le z \le L_{12}$are shown using each of the mentioned integration alternatives.

Table-1. Flexibility factors$0 \le z \le L_{12}$.

	Taylor	Gauss quadrature (3 points)*	Romberg
f11 (x10^{-4})	7.06817428	7.06820011	7.0682053
f22	0.05833378	0.05839166	0.05842843
f33 (x10^{-6})	2.31373921	2.31438778	2.31452246
f23 (x10^{-4})	3.26851004	3.27192528	3.27263719

As can be seen in Table-2, the values obtained from the integration of flexibility terms by the three methods are very similar. The approximations of Romberg and Gaussian Quadrature of three points are the most similar. For the purposes of the analysis of the problem was selected Gauss Quadrature to be typically more accurate in most cases.

The results of the fixed-end rotationend moments and the entries of the stiffness matrix of the tapered element, obtained from the integration of flexibility terms using Gauss Quadrature, are presented in Table- 2 and Table-3.

Table-2. Rotations and fixed moments of the tapered beam.

θ_{1x}	0.1117868
θ_{2x}	0.11404067
M_{1x}	24762.4553
M_{2x}	-24398.2844

Table-3. Terms of stiffness of the tapered element.

raz	433.8017001539121
r11x	590628.5173173111
r12x	361818.4266825598
r21x	361818.4266825598
r22x	568611.2477594916
rabx	1336.767640701573
rbax	1305.86620974323
raax	3.70895979009797

The values of the rotations and the reactions at the ends of the tapered element obtained from the model proposed in Maxima (GNU) and finite element models in *Ansys®and SAP2000®*, are compared in Table-4, verifying the efficiency of the method based on the numerical integrations of flexibility terms.

Table-4. Rotations and reactions at the ends of the beam.

	|θ1x|	|θ2x|	F1y	F2y
Maxima(GNU)	0.111787	0.114041	178.125	178.125
SAP2000®	0.108433	0.110898	178.125	178.125
Ansys®	0.111798	0.114025	178.125	178.125

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The coincidence between the results of the proposed model and Ansys® is evident, mentioning that the differences with *SAP2000*® are due to the fact that in its formulation the shear deformations are not taken into account, so the remaining deformation corresponds to the contribution of the effects of shear forcesin the calculation of the deformations that in some cases can be negligible (high L/h ratio, ≥ 10).

Figure-9 and Figure-10 present the results of the modeling of aT-beam with straight haunches, in the programs Ansys® and SAP2000® respectively, accounting for the effectiveness of the flexibility method implemented in the calculation program in Maxima (GNU) that considers bending and shear deformations.

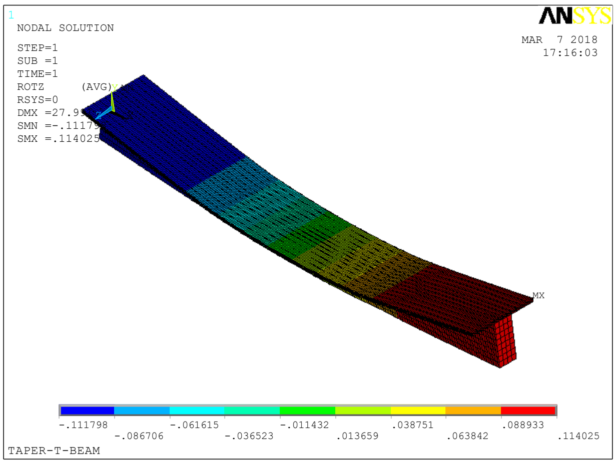

Figure-9. Displacement contoursof tapered T-beam in Ansys®.

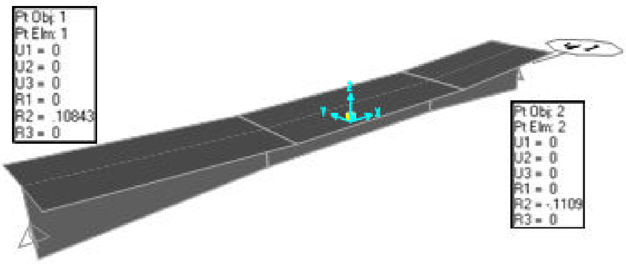

Figure-10. Results of the rotations at the ends tapered T-beamSAP2000®.

4. CONCLUSIONS

- The Matrix calculus system implanted in Maxima (GNU) based on the numerical integration of flexibility terms yields very similar deformation results to those of Ansys® for T-beams with linear haunches, which allows validating the proposed model.

- The differences in deformation results with SAP2000 ® is because the formulation used by this widely-used structural calculation package on a global scale does not take into account shear deformations.

- It was demonstrated that a systematic and reasonable standard-setting calculation methodology based on numerical integration can match the results obtained from robust finite element tools, making it a more economical and reliable option.

- We verified the effectiveness and precision of the developed model, representing significant advantages in the structural calculation of tapered beams with linearly-varying heights.

- It was demonstrated that the calculation of the stiffness matrix of variable section elements using matrix methods is viable using numerical integration due to the great development that the computational field has had.

- It was shown that with the calculation possibilities that the processors offer us today and with the advances in mathematical calculation tools, it is possible to solve problems of relevance in structural engineering without the help of expensive commercial finite element software.

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