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CHARACTERIZATION OF OUTER MEASURE OF PARTIAL LATTICES IN A COUNTABLE BOOLEAN LATTICE

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ABSTRACT

This paper describes that countable join (meet) of members of subsets of L, meet (join) of countable join (meet) of members of subsets of L, a measure on a Boolean lattice, outer measure of a partial lattice induced by a measure. Some properties of outer measure are verified and established that the measure and outer measure of any partial lattice in a Boolean lattice are equal and the compliment of partial lattice is also a partial lattice, also define measurability of any partial lattice in a countable Boolean lattice L. Finally verifies that the equivalent conditions of various forms of Littlewoods's first principle using $\mathcal{L}_\sigma$, $\mathcal{L}_\delta$, $\mathcal{L}_{\sigma\delta}$, $\mathcal{L}_{\delta\sigma}$.

Keywords: Boolean lattice, little Woods's first principle, partial lattice.

1. INTRODUCTION

[1] In a distributive lattice L If there exists a unique complement l′ such that l∨ l′ = 1(one) and l ∧ l′ = 0 (zero) for any element l in L then L is said to be Boolean Lattice. If a Boolean lattice L is closed under countable joins then L is called countable Boolean Lattice. That is any sequence $\{l_n\}$ in L has l.u.b. in L. [3] A nonempty subset H of a countable Boolean lattice L is called a partial lattice if for $h_1, h_2, h_3 \in H$, if $h_1 \wedge h_2 = h_3$ (dually, $h_1 \vee h_2 = h_3$), then we say that $h_1 \wedge h_2$ (dually $h_1 \vee h_2$) is defined in H and it equals h_3. If for $h_1, h_2 \in H$, $h_1 \wedge h_2 \notin H$ (duall $h_1 \vee h_2 \notin H$), then we say that $h_1 \wedge h_2$ (dually $h_1 \vee h_2$) are not defined in H. Thus (H, ∧, ∨) is a set with two partial binary operations. (H, ∧, ∨) is called a partial lattice or a relative sublattice of L.

In section2 we define countable join (meet) of members of subsets of L, meet (join) of countable join (meet) of members of subsets of L, a measure on a Boolean lattice, outer measure of a partial lattice induced by a measure.

In section3 the properties of outer measure are verified and establish results that the measure and outer measure of any partial lattice in a Boolean lattice are equal, the compliment of the partial lattice is also a partial lattice.

In section4 we define a measurability of any partial lattice in a countable Boolean lattice L. Also verify equivalent conditions of various forms of Littlewoods's first principle using $\mathcal{L}_\sigma$, $\mathcal{L}_\delta$, $\mathcal{L}_{\sigma\delta}$, $\mathcal{L}_{\delta\sigma}$.

2. PRELIMINARIES

Let L be a countable Boolean lattice and $\mathcal{L}$ be a Boolean lattice of partial lattices of L under the set inclusion.

Definition 2.1 For any set $\mathcal{A}$ of subsets of L, $\mathcal{A}_\sigma$ stands for the collection of countable join of members of $\mathcal{A}$ and $\mathcal{A}_\delta$ stands for the collection of countable meet of members of $\mathcal{A}$. $\mathcal{A}_{\sigma\delta}$ stands for the join of collection of countable meet of members of $\mathcal{A}$, $\mathcal{A}_{\delta\sigma}$ stands for the meet of collection of countable join of members of $\mathcal{A}$.

Example 2.1

(1) If $\mathcal{A} = P(\mathbb{R})$ then $\mathcal{A}_\sigma = \mathcal{A}_\delta = P(\mathbb{R})$

(2) $L = \{(a, \infty) / a \in \mathbb{R}\} \cup \mathbb{R}$ is $\mathcal{L}_\sigma = L \cup \{\phi\}$

Let L be a lattice and $\mathcal{L}$ a collection of partial lattices in L closed under join and meet and containing 0 and L.

Definition 2.2 (Measure on a Boolean lattice) [2] A non-negative extended real valued function ɲ defined on $\mathcal{L}$ such that (1) ɲ(0) = 0. (2) If $\{A_n\}$ is any sequence of pairwise disjoint elements in L then $ɲ(\bigvee_{i=1}^{\infty} A_n) = \sum_{i=1}^{\infty} ɲ(A_n)$ whenever $\bigvee_{i=1}^{\infty} A_n \in \mathcal{L}$ is called measure on $\mathcal{L}$.

Remark 2.1. If $\mathcal{L}$ is a countable Boolean lattice then $\bigvee_n a_n \in \mathcal{L}$ for every sequence $\{a_n\} \in \mathcal{L}$.

Definition 2.3 If ɲ is a measure on (L, $\mathcal{L}$), for any partial lattice H ⊂ L.

Define $ɲ^*(H) = \text{Inf}\ \{ \sum_{i=1}^{\infty} ɲ(A_i) / H \subset \bigvee_{i=1}^{\infty} A_i$ and $A_i \in \mathcal{L}$ } ɲ* is called the outer measure induced by ɲ.

Remark 2.2 In this context, the measure is defined on elements of the Boolean lattice of partial lattices in L, whereas the outer measure is defined for any partial lattice H in L.

3. PROPERTIES OF ɲ*

Properties of ɲ*

(1) ɲ*(0) = 0

(2) ɲ*(H) ≥ 0 ∀ partial lattices H

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(3) $ɲ^*(\bigvee_{i=1}^{\infty} H_i) \le \sum_{i=1}^{\infty} ɲ^*(H_i)$ for every countable sequence of partial lattices $\{H_i\}$ in L.

Proof of (3): We may assume that $ɲ^*(H_i) < \infty\ \forall$ i

Fix i, there exist a sequence $\{H_{i,n}\}$, n = 1,2,….. in L such that $H_i \subset \bigvee_{i=1}^{\infty} H_{i,n}$ and

$\sum_{i=1}^{\infty} ɲ(H_{i,n}) \le ɲ^*(H_i) + \frac{\varepsilon}{2^i} \forall$ i.

Clearly $\{H_{i,n}\}$ for I ≥ 1, n ≥ 1 is countable collection of partial lattices in $\mathcal{L}$ and $H_i \subset \bigvee_{i,n} H_{i,n}$,

Further $ɲ^*(\bigvee_{i=1}^{\infty} H_i) \le \sum_{i,n} ɲ(H_{i,n}) \le \sum_{i} ɲ^*(H_i) + \frac{\varepsilon}{2^i} = \sum ɲ^*(H_i) + \varepsilon$ this true for every $\varepsilon > 0$

So $ɲ^*(\bigvee_{i=1}^{\infty} H_i) \le \sum_{i=1}^{\infty} ɲ^*(H_i)$

Result 3.1 $ɲ^*(A) = ɲ(A)$ if $A \in \mathcal{L}$.

Proof: Clearly $A \subseteq A$

If $A \in \mathcal{L}$, $ɲ^*(A) \le ɲ(A)$ (1)

Given $\varepsilon > 0$ there is sequence $\{A_n\}$ in L such that $A \subset \vee A_n$ and $\sum ɲ(A_n) < ɲ^*(A) + \varepsilon$

Since $A \subset \vee A_n$, $ɲ(A) \le ɲ(\vee A_n)$

$\le \sum ɲ(A_n)$ [By countable subadditive property[2]]

$< ɲ^*(A) + \varepsilon$

This is true for every $\varepsilon > 0$

Therefore $ɲ(A) \le ɲ^*(A)$ (2)

From (1) and (2) $ɲ^*(A) = ɲ(A)$.

Result 3.2 If H is a partial lattice then $H' = \{a' / a \in H\}$ is a partial lattice.

Proof: Let $b', c', d' \in H'$, if $b' \wedge c' = d'$, then $(b \vee c)' = d'$

Implies $b \vee c = d \in H$. Therefore $b' \wedge c' =$ glb $\{b', c'\}$ in H'

Dually, if $b' \vee c' = d'$, then $(b \wedge c)' = d'$. Implies $b \wedge c = d \in H$

Therefore $b' \vee c' =$ lub $\{b', c'\}$ in H'. Hence H' is a partial lattice.

4. MEASURABILITY OF PARTIAL LATTICES

Let L be a countable Boolean lattice, $\mathcal{L}$ be a Boolean lattice of partial lattices in L. Clearly $H \in \mathcal{L}$ implies $H^c = L - H \in \mathcal{L}$. We now define measurability of a partial lattice $A \subset L$ via a given measure on $\mathcal{L}$

Definition 4.1 A partial lattice $A \subset L$ is said to be $ɲ^*$ measurable if for any $E \in \mathcal{L}$, $ɲ^*(E) = ɲ^*(E \wedge A) + ɲ^*(E \wedge A^c)$.

Note 4.1 For convenience, we write measurable for $ɲ^*$ measurable.

We assume the following Littlewoods's condition for ɲ:

If $A \subset L$ and for every $\Delta > ɲ^*(A)$, there exist a sequence of disjoint partial lattices $\{A_n\}$ in $\mathcal{L}$ such that $A \subseteq \vee A_n$ and $\sum ɲ^*(A_n) < \Delta$.

Theorem 4.1 Let A be a partial Lattice and $\varepsilon > 0$, there is a sequence $\{A_n\}$ in $\mathcal{L}$ and $\varepsilon > 0$ there is O in $\mathcal{L}_\sigma$ such that $A \subseteq \mathcal{L}_\sigma$ and $ɲ^*(O) \le ɲ^*(A) + \varepsilon$. There is $\mathcal{L}_{\sigma\delta}$ partial lattice G such that $A \subseteq G$ and $ɲ^*(A) = ɲ^*(G)$.

Proof: Let A be any partial Lattice.

Part - 1:

Let $\varepsilon > 0$

First, we show that there is O in $\mathcal{L}_\sigma$ such that $A \subseteq O$ and $ɲ^*(O) \le ɲ^*(A) + \varepsilon$.

Case (i).Suppose $ɲ^*(A) = \infty$.

Take the O = A.

Clearly $A \subseteq O$ and $ɲ^*(O) = \infty$.

Also $ɲ^*(A) + \varepsilon = \infty$

Implies $ɲ^*(O) \le ɲ^*(A) + \varepsilon$.

Case (ii).Suppose $ɲ^*(A)$ is finite. That is $ɲ^*(A) < \infty$.

By Littlewoods's property, for every $\varepsilon > ɲ^*(A)$, there exist a sequence of disjoint partial lattices $\{A_n\}$ in $\mathcal{L}$ such that $A \subseteq \vee A_n$ and $\sum ɲ^*(A_n) < \varepsilon$.

Take $O = \bigvee_n A_n$ Clearly $A \subseteq O$

Also $ɲ^*(O) \le ɲ^*(\bigvee_n A_n) \le \sum_n ɲ^*(A_n) < ɲ^*(A) + \varepsilon$

Part - 2:

Now we will show that there is G in $\mathcal{L}_{\sigma\delta}$ lattice such that $A \subseteq G$ and $ɲ^*(A) = ɲ^*(G)$.

Case (i).Suppose $ɲ^*(A) = \infty$. Take $A_n = A\ \forall$ n and $G = \wedge A_n$

Clearly, G is in $\mathcal{L}_{\sigma\delta}$ lattice such that $A \subseteq G$ and also $ɲ^*(A) = \infty = ɲ^*(G)$.

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Case (ii): Suppose $\mu^*(A) < \infty$. For any $\varepsilon > 0$, by part – 1, there exist a partial lattice A_n in $\mathcal{L}_\sigma$ such that $A \subseteq A_n$ and $\mu^*(A_n) \leq \mu^*(A) + \frac{1}{n}$. This is true $\forall$ n.

Take $G = \wedge_n A_n$. Clearly, G is in $\mathcal{L}_{\sigma\delta}$ and $A \subseteq G$

Implies $\mu^*(A) \leq \mu^*(G)$ (3)

We have $G \subseteq A_n \; \forall$ n. Implies $\mu^*(G) \leq \mu^*(A_n) \leq \mu^*(A) + \frac{1}{n} \; \forall$ n.

Implies $\mu^*(G) \leq \mu^*(A)$ (4)

From (1) and (2) we have
$\mu^*(A) = \mu^*(G)$.

Result 4.1 If B_1 and B_2 are measurable partial lattices then $B_1 - B_2$ and $B_1 \Delta B_2$ are also measurable partial lattices.

Proof: Suppose B_2 is measurable partial lattice
Implies B_2^c is measurable partial lattice
Implies $B_1 \wedge B_2^c$ is measurable partial lattice
Now $B_1 \wedge B_2^c = B_1 - B_2$
But $B_1 \wedge B_2^c$ is measurable partial lattice
Implies $B_1 - B_2$ is measurable partial lattice
Similarly, $B_2 - B_1$ is measurable partial lattice

Now, $B_1 \Delta B_2 = (B_1 - B_2) \vee (B_2 - B_1)$ is also a measurable partial lattice.

Theorem 4.2 Suppose E is any partial lattice in $\mathcal{L}$. Then the following five statements are equivalent.

a) E is measurable.

b) Given $\varepsilon > 0$ there is partial lattice O in L_σ such that O contains E and $\mu^*(O - E) < \varepsilon$.

c) Given $\varepsilon > 0$ there is partial lattice F in L_δ such that $F \subseteq E$ and $\mu^*(E - F) < \varepsilon$.

d) There is partial lattice G in $L_{\sigma\delta}$ with $E \subseteq G$ and $\mu^*(G - E) = 0$.

e) There is partial lattice F in $L_{\delta\sigma}$ with $F \subseteq E$ and $\mu^*(E - F) = 0$.

Proof:
Part - 1:
To prove that (1) implies (2)
Case (i) suppose E is measurable and $\mu^*(E) < \infty$.

By above theorem4.1, there exist partial lattice O in $\mathcal{L}_\sigma$ such that $E \subseteq \mathcal{L}_\sigma$ and $\mu^*(O) \leq \mu^*(E) + \varepsilon$
Since E is measurable,
$\mu^*(O) = \mu^*(O \wedge E) + \mu^*(O \wedge E^c)$
Implies $\mu^*(O) = \mu^*(E) + \mu^*(O - E)$
Implies $\mu^*(O - E) = \mu^*(O) - \mu^*(E) < \varepsilon$
Therefore $\mu^*(O - E) < \varepsilon$

Hence for given $\varepsilon > 0$, there exist $O \in \mathcal{L}_\sigma$ such that $O \supset E$ and $\mu^*(O - E) < \varepsilon$, provided $\mu^*(E) < \infty$
Case (ii) supposes $\mu^*(E)$ is infinite.

Let $O = \vee_n A_n$, where $\{A_n\}$ is a countable collection of disjoint partial lattices.

Now $E = E \wedge O = E \wedge (\vee_n A_n) = \vee_n (E \wedge A_n) = \vee_n E_n$ where $E_n = E \wedge A_n \; \forall$ n

Clearly, each E_n is measurable and $\mu^*(E_n) \leq \mu^*(A_n) \; \forall$ n.

By case (i), there exist A_n in $\mathcal{L}_\sigma$, $A_n \supseteq E_n$ such that $\mu^*(A_n - E_n) < \varepsilon$

Since $A_n \supseteq E_n \; \forall$ n implies $\vee_n A_n \supset \vee_n E_n$ implies $O \supset E$

Now $O - E = \vee_n A_n - \vee_n E_n \subseteq \vee_n (A_n - E_n)$
$\mu^*(O - E) \leq \mu^*[\vee_n (A_n - E_n)] \leq \sum_n \mu^*(A_n - E_n) < \varepsilon$
Therefore, $\mu^*(O - E) < \varepsilon$

Therefore for given $\varepsilon > 0$ there exist O in $\mathcal{L}_\sigma$, $O \supseteq E$ such that $\mu^*(O - E) < \varepsilon$
Hence (1) implies (2).
Part - 2:
To prove that (2) implies (4)
Assume (2).

That is; for a given $\varepsilon > 0$, there exist O in $\mathcal{L}_\sigma$, $O \supseteq E$ such that $\mu^*(O - E) < \varepsilon$.

By (2), for any positive integer n, there exist partial lattice A_n in $\mathcal{L}_\sigma$, $A_n \supseteq E$ such that $\mu^*(A_n - E) < \frac{1}{n} = \varepsilon$.
Write $G = \wedge_n A_n$.
Then clearly G is in $\mathcal{L}_{\sigma\delta}$

Since $E \subseteq A_n \; \forall$ n,
$E \subset \wedge_n A_n$ Implies $E \subseteq G$
Now $G \subseteq A_n \; \forall$ n
Implies $G - E \subseteq A_n - E \; \forall$ n
Implies $\mu^*(G - E) \leq \mu^*(A_n - E) < \varepsilon, \forall$ n
Implies $\mu^*(G - E) = 0$.

Therefore there exist partial lattice G in $\mathcal{L}_{\sigma\delta}$ such that $E \subseteq G$ and $\mu^*(G - E) = 0$

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Hence (2) implies (4).
PART - 3:
To prove that (4) implies (1)
Assume (4).

That is; there is G in $\mathcal{L}_{\sigma\delta}$ such that E ⊆G and ɲ*(G – E) = 0.
Now G is measurable, E is measurable.
So G – E is measurable.

Since G is in $\mathcal{L}_{\sigma\delta}$, we can write G as a countable intersection of partial lattices in $\mathcal{L}_{\sigma}$
Therefore G is measurable.
Now E = G – (G – E) is measurable.
Therefore E is measurable. [By result 4.1]
Hence (4) implies (1).
PART – 4:
To prove that (1) implies (3)
Assume (1).
That is; E is measurable.
By the Result3.2, E^c is measurable.

Let ε> 0, by (2) there exist $\mathcal{L}_{\sigma}$ lattice O containing E^c such that ɲ*(O – E^c) <ε.
Write F = O^c [since O ⊇ E^c Implies O^c ⊆ E Implies F ⊂E]
Consider ɲ*(E – F) = ɲ*(E – O^c)
= ɲ*(E ∧ O) [since A – B = A ∧ B^c]
= ɲ*(O ∧ E) [By commutative law]
= ɲ*(O – E^c) [since A – B = A ∧ B^c]
<ε

Therefore for given ε> 0 there exist a partial lattice F in $\mathcal{L}_{\delta}$, F ⊆ E and ɲ*(E – F) <ε
Hence (1) implies (3).
PART - 5:
To prove that (3) implies (5).
Assume (3).

That is; for a given ε> 0, there exist a partial lattice F in $\mathcal{L}_{\delta}$ such that F ⊆E and ɲ*(E – F) <ε.

In particular for all positive integer n, there exist a partial lattice F_n in $\mathcal{L}_{\delta}$, F_n ⊆E such that ɲ*(E – F_n) < $\frac{1}{n}$.
Write F = $\vee_n F_n$
Then F is in $\mathcal{L}_{\delta\sigma}$ and also F ⊆E.
[since F_n ⊆E ∀ n, implies $\bigvee_{n=1}^{\infty} F_n \subseteq E$. Implies F ⊆ E]
Consider 0 ≤ ɲ*(E – F) ≤ ɲ*(E – F_n) < $\frac{1}{n}$ ∀ n.
This is true for all n.
Implies ɲ*(E – F) = 0

Therefore there exist a partial lattice F in $\mathcal{L}_{\delta\sigma}$, F ⊆E such that ɲ*(E – F) = 0.
PART - 6:
To prove that (5) implies (1)
Assume (5).

That is; there is a partial lattice F in $\mathcal{L}_{\delta\sigma}$, F ⊆E such that ɲ*(E – F) = 0.
Implies E - F is measurable [By result4.1.]

Since F is in $\mathcal{L}_{\delta\sigma}$, we can write F as a countable join of partial lattices in $\mathcal{L}_{\delta}$
That is F = $\vee_n F_n$. We know that each F_n is measurable.
Therefore F is measurable.
Now E = F ∨ (E – F) is measurable.
Therefore (5) implies (1).

Hence (1) implies (2) implies (3) implies (4) implies (5).

5. CONCLUSIONS

We defined countable join (meet) of members of subsets of L, meet (join) of countable join (meet) of members of subsets of L, a measure on a Boolean lattice, outer measure of a partial lattice induced by a measure. The properties of outer measure are verified and established that the measure and outer measure of any partial lattice in a Boolean lattice are equal and the compliment of partial lattice is also a partial lattice, also defined measurability of any partial lattice in a countable Boolean lattice L. Finally verified that the equivalent conditions of various forms of Littlewoods's first principle using $\mathcal{L}_{\sigma}$, $\mathcal{L}_{\delta}$, $\mathcal{L}_{\sigma\delta}$, $\mathcal{L}_{\delta\sigma}$.

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