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NUMERICAL SOLUTION OF NEUTRON DENSITY USING THE EXPLICIT THIRD ORDER, THIRD STAGE STOCHASTIC RUNGE-KUTTA METHOD

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ABSTRACT

This paper presents the explicit third order, third stage stochastic Runge-Kutta method for providing a solution to stochastic point kinetic equations for various forms of reactivity, considering one and six groups of delayed neutron precursors. Point kinetic equations are a set of non-linear stochastic differential equations, which are resolved by the method proposed in each instant of time. Calculation is made of the mean values and the standard deviation of the neutron density and the concentration of delayed neutron precursors in the proposed numerical experiments. The results obtained are compared with other methods reported in literature, confirming that the method is efficient and accurate for studying the dynamics of a point nuclear reactor.

Keywords: stochastic point kinetic, concentration of delayed neutron precursors, neutron density, stochastic Runge-Kutta method.

INTRODUCTION

Due to the increase in world population and technological developments, it has become necessary to find new sources of efficient energy with low environmental impact, and one of the sources with these characteristics is nuclear energy. The place where the generation of nuclear energy takes place is the nuclear power reactor [1]. This device allows the starting, maintenance and control of the fission reactions of the fuel atoms. Point kinetic equations [2] study the dynamics of nuclear reactors based on the analysis of the behavior of neutrons produced in the reactor core by means of nuclear reactivity.

The operability of a nuclear reactor is stochastic by nature [3]. This characteristic is evident in critical and subcritical nuclear reactors, and the density of neutrons and the concentration of delayed neutron precursors show random fluctuations over time, therefore point kinetic equations must be described by a stochastic process. This study performs an analysis of the stochastic point kinetic equations, and carries out various numerical experiments to establish the mean and the variance of the density of neutrons and the concentration of the delayed neutron precursors from the numerical approximations generated by the third order, third stage stochastic Runge-Kutta (RK3-3st) method [4].

In this study, first of all a theoretical review related to point kinetics is conducted. A description follows of the explicit RK3-3st method, a number of numerical experiments are then made to validate the proposed method, and finally the conclusions of the research are presented.

THEORETICAL ASPECTS

The equation of the deterministic kinetics of a nuclear reactor is given by [5]:

$$\frac{\partial N(r,t)}{\partial t} = D\upsilon\nabla^2 N(r,t) - \left(\sum_a - \sum_f\right)\upsilon N(r,t) + \left[(1-\beta)K_\infty \sum_a - \sum_f\right]\upsilon N(r,t) + \sum_{i=1}^{m}\lambda_i C_i(r,t) + S_0(r,t) \quad (1)$$

$$\frac{\partial C_i(r,t)}{\partial t} = \beta_i K_\infty \sum_a \upsilon N(r,t) - \lambda_i C_i(r,t) \quad (2)$$

Equation (1) corresponds to the neutron diffusion equation, and equation (2) corresponds to the concentration of delayed neutron precursors. Where $i = 1, 2, ..., m$ is the group number of delayed neutron precursors, $N(r,t)$ is the neutron density, $C_i(r,t)$ is the concentration of i-th group of delayed neutron precursors, υ is the velocity of fission neutrons. The term $D\upsilon\nabla^2 N(r,t)$ explains the diffusion of neutrons, $\sum_a$ is the transversal section of absorption , $\sum_f$ is the transversal section of fission , $\sum_a - \sum_f$ is the transversal section of capture, $1-\beta$ is the prompt-neutron fraction, the contribution of neutrons to the source is given by the term $\left[(1-\beta)K_\infty \sum_a - \sum_f\right]\upsilon N(r,t)$, β_i is the fraction of delayed neutrons of the i-th group of precursors, β is the total fraction of delayed neutrons, K_∞ is the multiplication factor in an infinite medium; it is the rate of transformation of neutron precursors to the neutron

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population, λ_i is the decay constant of the i-th group of precursors and $S_0(r,t)$ is the neutron source.

Supposing that the density of neutrons and the concentration of precursors are separable functions in time, and space

$$N(r,t) = f_{(r)} n_{(t)} \tag{3}$$

$$C_i(r,t) = g_{(r)} c_{i(t)} \tag{4}$$

where $n_{(t)}$ is the density of neutron population and $c_{i(t)}$ is the density of the population of the i-th group of delayed neutron precursors.

Evaluating the considerations of equations (3) and (4) in equation (1) the following is obtained

$$\frac{\partial n_{(t)}}{\partial t} = D\upsilon \frac{\nabla^2 f_{(r)}}{f_{(r)}} n_{(t)} - \left(\sum_a - \sum_f \right) \upsilon n_{(t)} + \left[(1-\beta) K_\infty \sum_a - \sum_f \right] \upsilon n_{(t)} + \sum_{i=1}^{m} \lambda_i \frac{g_{(r)}}{f_{(r)}} c_i + \frac{S_0(r,t)}{f_{(r)}} \tag{5}$$

$$\frac{\partial c_{i(t)}}{\partial t} = \beta_i K_\infty \sum_a \upsilon \frac{f_{(r)}}{g_{(r)}} n_{(t)} - \lambda_i c_{i(t)} \tag{6}$$

The spatial density of neutrons is maintained harmonically, and is described by a Helmholtz equation $\nabla^2 f_{(r)} + B_0^2 f_{(r)} = 0$ where B_0^2 refers to geometric buckling, considering that $\frac{g_{(r)}}{f_{(r)}} = 1$ due to the fact that $\frac{g_{(r)}}{f_{(r)}}$ is independent of time, taking into account that $S_0(r,t)$ has the same spatial dependence as $f_{(r)}$ it is proposed that $q_0 = \frac{S_0(r,t)}{f_{(r)}}$. On replacing equation (6) in (5) and taking into account the proposed considerations, we have

$$\frac{\partial n_{(t)}}{\partial t} = -D\upsilon B_0^2 n_{(t)} - \left(\sum_a - \sum_f \right) \upsilon n_{(t)} + \left[(1-\beta) K_\infty \sum_a - \sum_f \right] \upsilon n_{(t)} + \sum_{i=1}^{m} \lambda_i c_i + q_{(t)} \tag{7}$$

$$\frac{\partial c_{i(t)}}{\partial t} = \beta_i K_\infty \sum_a \upsilon n_{(t)} - \lambda_i c_{i(t)} \tag{8}$$

The neutron density equation given by equation (7). The expression of absorption lifetime is defined $l_\infty = \frac{1}{\upsilon \sum_a}$, diffusion length $L^2 = \frac{D}{\sum_a}$, multiplication factor $k = \frac{K_\infty}{L^2 B_0^2 + 1}$, the neutron lifetime $l_0 = \frac{l_\infty}{L^2 B_0^2 + 1}$, the generation time $\Lambda = \frac{l_0}{k}$, also considering $\alpha = \frac{\sum_f}{K_\infty \Lambda \sum_a} \approx \upsilon$ and the reactivity function is established as $\rho_{(t)} = 1 - \frac{1}{k}$. Replacing these parameters in equations (7) and (8), we obtain

$$\frac{\partial n_{(t)}}{\partial t} = \left[\frac{\rho_{(t)} - \beta}{\Lambda} \right] n_{(t)} + \sum_{i=1}^{m} \lambda_i c_i + q_{(t)} \tag{9}$$

$$\frac{\partial c_{i(t)}}{\partial t} = \frac{\beta_i}{\Lambda} n_{(t)} - \lambda_i c_{i(t)} \tag{10}$$

Equation (9) corresponds to neutron density, and equation (10) is the concentration of delayed neutron precursors. Now, equations (9) and (10) are considered as a very small interval of time Δt; in this time interval, the neutrons in the reactor can present four possible events in the behavior of the population. These are: capture, fission, transformation of a delayed neutron precursor, and the birth of a source neutron. This change in the neutron population and the concentration of the i-th group of precursors for each one of the events is given by $|\Delta \hat{x}\rangle = \left[\Delta n_{(t)}, \Delta c_{1(t)}, \Delta c_{2(t)}, \cdots, \Delta c_{m(t)} \right]^T$. The events with their respective probabilities are shown in Table-1.

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Table-1. Possible neutron events in the reactor.

Event	Change of neutron population $\left[\Delta n_{(t)}\right]_k$	Change of i-th group of precursors population $\left[\Delta c_{i(t)}\right]_k$	Probability P_k
Capture	-1	0	$dn_{(t)}\Delta t$
Fission	$-1+(1-\beta)\upsilon$	$\beta_i\upsilon$	$bn_{(t)}\Delta t$
Transformation	1	-1	$\lambda_i c_{i(t)}\Delta t$
Birth	1	0	$q_{(t)}\Delta t$

The mean change and variance of the neutron population rate in the small time interval is given by

$$E\left(\left|\Delta\hat{x}\right\rangle\right)=\sum_{k=1}^{4}P_k\left|\Delta\hat{x}\right\rangle_k=\begin{bmatrix}\left(\frac{\rho_{(t)}-\beta}{\Lambda}\right)n_{(t)}+\sum_{i=1}^{m}\lambda_i c_{i(t)}+q_{(t)}\\ \frac{\beta_i}{\Lambda}n_{(t)}-\lambda_1 c_{1(t)}\\ \frac{\beta_2}{\Lambda}n_{(t)}-\lambda_2 c_{2(t)}\\ \vdots\\ \frac{\beta_m}{\Lambda}n_{(t)}-\lambda_m c_{m(t)}\end{bmatrix}\Delta t \quad (11)$$

$$\sigma\left(\left|\Delta\hat{x}\right\rangle\right)=\sum_{k=1}^{4}P_k\left|\Delta\hat{x}\right\rangle_k\left\langle\Delta\hat{x}\right|_k=B\Delta t \quad (12)$$

In equation (12) B is the covariance matrix which is defined as

$$B=\begin{bmatrix}\zeta & a_1 & a_2 & \cdots & a_1\\ a_1 & r_1 & b_{1,2} & \cdots & b_{1,m}\\ a_2 & b_{2,1} & r_2 & \cdots & b_{2,m}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_m & b_{m,1} & b_{m,2} & \cdots & r_m\end{bmatrix} \quad (13)$$

Where

$$\zeta=\left[\frac{-1-\rho_{(t)}+2\beta+(1-\beta)^2\upsilon}{\Lambda}\right]n_{(t)}+\sum_{i=1}^{m}\lambda_i c_{i(t)}+q_{(t)}$$

$$a_i=\frac{\beta_i}{\Lambda}\left[(1-\beta)\upsilon-1\right]n_{(t)}-\lambda_i c_{i(t)}$$

$$b_{i,j}=\frac{\beta_i\beta_j}{\Lambda}n_{(t)}$$

$$r_i=\frac{\beta_i\beta_i}{\Lambda}n_{(t)}+\lambda_i c_{i(t)}$$

Calculating the central limit theorem and considering that the changes in the time interval Δt are distributed in approximately the normal manner, we obtain [6]

$$d\hat{P}_{(t)}=\left[A\hat{x}_{(t)}+\hat{F}_{(t)}\right]dt+B^{1/2}d\hat{W}_{(t)} \quad (14)$$

Where

$$A=\begin{bmatrix}\frac{\rho_{(t)}-\beta}{\Lambda} & \lambda_1 & \lambda_2 & \cdots & \lambda_m\\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & \cdots & 0\\ \frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ \frac{\beta_m}{\Lambda} & 0 & 0 & \cdots & -\lambda_m\end{bmatrix} \quad (15)$$

$\hat{P}_{(t)}$, $\hat{F}_{(t)}$ and $\hat{W}_{(t)}$ are random variable vectors and are defined as

$$\hat{P}_{(t)}=\begin{bmatrix}n_{(t)}\\ c_{1(t)}\\ c_{2(t)}\\ \vdots\\ c_{m(t)}\end{bmatrix};\ \hat{F}_{(t)}=\begin{bmatrix}q_{(t)}\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix};\ \hat{W}_{(t)}=\begin{bmatrix}W_{1(t)}\\ W_{2(t)}\\ W_{3(t)}\\ \vdots\\ W_{1(t)}\end{bmatrix} \quad (16)$$

The covariance matrix B was defined in equation (13), and $\hat{W}_{(t)}$ is the Weiner process or Brownian movement defined in the time interval $[t_0,T]$. In literature there are reports of different methods which offer a solution to stochastic point kinetic equations. These methods are: Piecewise Constant Approximation(PCA) and Monte Carlo [6], Euler-Maruyama (EM), Taylor 1.5 [3, 7], Simplificated Stochastic Point Kinetics (SSPK) [8], Analytical Exponential Model (AEM) [9], Efficient stochastic model (ESM) [10], Double-Diagonal and Decomposition Method (Double DDM) [11] and the Euler-Maruyama implicit method (EM implicit) [12].

PROPOSED METHOD

In this section, we will present the stochastic Runge-Kutta method of stage s for the scalar case [4], represented in the form

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$$\bar{X}_{n+1} = \bar{X}_n + \Delta\sum_{j=1}^{s}\alpha_j a\left(t_n+\mu_j\Delta,\eta_j\right) + \Delta\hat{W}_n\sum_{j=1}^{s}\beta_j b\left(t_n+\mu_j\Delta,\mu_j\right) + R \quad (17)$$

where $\mu_1 = 0$, $\eta_1 = \bar{X}_n$ and

$$\eta_j = \bar{X}_n + \Delta\sum_{i=1}^{j-1}\lambda_{ji} a\left(t_n+\mu_i\Delta,\eta_i\right) + \Delta\hat{W}_n\sum_{i=1}^{j-1}\gamma_{ji} b\left(t_n+\mu_i\Delta,\eta_i\right),$$
$$j = 2,...,s$$

$\alpha_j, \mu_j, \beta_j, \lambda_{ji}$ and γ_{ji} are numerical constants, R is an appropriate term and $\Delta\hat{W}_n$ are Wiener processes. To deduce the explicit third order Runge-Kutta scheme $s = 3$ is considered in equation (17). Then,

$$\bar{X}_{n+1} = \bar{X}_n + \left[\alpha_1 a + \alpha_2 a\left(t_n+\mu_2\Delta,\eta_2\right) + \alpha_3 a\left(t_n+\mu_3\Delta,\eta_3\right)\right]\Delta + \beta b\Delta\hat{W}_n + R \quad (18)$$

Where

$$\eta_2 = \bar{X}_n + \lambda_{21} a\Delta + \gamma_2 b\Delta\hat{W}_n \quad ; \quad a = a\left(t_n,\bar{X}_n\right) \quad ; \quad b = b\left(t_n,\bar{X}_n\right)$$
$$\eta_3 = \bar{X}_n + \lambda_{31} a\Delta + \lambda_{32} a\left(t_n+\mu_2\Delta,\eta_2\right)\Delta + \gamma_3 b\Delta\hat{W}_n$$

It is worth noting that the terms $a\left(t_n+\mu_2\Delta,\eta_2\right)$ and $a\left(t_n+\mu_3\Delta,\eta_3\right)$ are functions which can be obtained as third order truncated expansions in the manner:

$$f\left(t_n+\Delta,X_t+\Delta X\right) \overset{(3)}{=} f + f_{10}\Delta + f_{01}\Delta X + \left(f_{20} - \frac{1}{4}b^4 f_{04}\right)\frac{\Delta^2}{2} + f_{11}\Delta\left(\Delta X\right) + \frac{1}{2}f_{02}\left(\Delta X\right)^2 +$$
$$\frac{1}{6}\left(f_{03} + \frac{3}{2}b^2 f_{22} + \frac{3}{4}b^4 f_{14} + \frac{1}{8}b^6 f_{06}\right)\Delta^3 + \left(f_{21} + b^2 f_{13} + \frac{1}{4}b^4 f_{05}\right)\frac{\Delta^2\left(\Delta X\right)}{2} + \quad (19)$$
$$\left(f_{12} + \frac{1}{2}b^2 f_{04}\right)\frac{\Delta\left(\Delta X\right)^2}{2} + f_{03}\frac{\left(\Delta X\right)^3}{6}$$

With $f_{ij} = \dfrac{\partial^{i+j} f\left(t_n,\bar{X}_n\right)}{\partial t^i \partial\bar{X}^j}$ and $\overset{(3)}{=}$ refers to the third order approximation of the function. Based on equation (19) and bearing in mind that the lineal combination of products of form $\Delta^i\left(\Delta\hat{W}_n\right)^j$ with $i = j = 1,2,...,n$ for the third order approximation corresponds to

$$\Delta\left(\Delta\hat{W}_n\right)^3 \overset{(3)}{=} 3\Delta^2\left(\Delta\hat{W}_n\right)$$
$$\Delta^2\left(\Delta\hat{W}_n\right)^2 \overset{(3)}{=} \Delta^3 \quad (20)$$
$$\Delta^i\left(\Delta\hat{W}_n\right)^j \overset{(3)}{=} 0 \quad \text{if} \quad i + \frac{j}{2} \geq \frac{7}{2}$$

The third order truncated expansion for functions $a\left(t_n+\mu_2\Delta,\eta_2\right)\Delta$ and $a\left(t_n+\mu_3\Delta,\eta_3\right)\Delta$, are given by:

$$a\left(t_n+\mu_2\Delta,\eta_2\right)\Delta \overset{(3)}{=} a\Delta + \left[a_{10}\mu_2 + \lambda_{21}a_{01}a\right]\Delta^2 + a_{01}\gamma_2 b\Delta\left(\Delta\hat{W}\right) +$$
$$\frac{1}{2}\left[\gamma_2^2 b^2 a_{02} + 2\mu_2\gamma_2 a_{11}b + 2\lambda_{21}\gamma_2 a_{02}ab + \gamma_2^3 b^3 a_{03}\right]\Delta^2\left(\Delta\hat{W}_n\right) \quad (21)$$
$$+\frac{1}{2}\left[2\lambda_{21}\mu_2 a_{11}a + \lambda_{21}^2 a_{02} + \left(a_{20} - \frac{1}{4}b^4 a_{04}\right)\mu_2^2 + \left(a_{12} + \frac{1}{2}b^2 a_{04}\right)\gamma_2^2\mu_2 b^2 + \lambda_{21}\gamma_2^2 a_{03}ab^2\right]\Delta^3$$

$$a\left(t_n+\mu_3\Delta,\eta_3\right)\Delta \overset{(3)}{=} a\Delta + a_{01}\gamma_3 b\Delta\left(\Delta\hat{W}_n\right) + \left(\lambda_{31}+\lambda_{32}\right)a_{01}a\Delta^2 +$$
$$\frac{1}{2}\left(a_{12} + \frac{b^2}{2}a_{04}\right)\mu_3\gamma_3^2 b^2\Delta^2\left(\Delta\hat{W}_n\right)^2 + \frac{1}{2}\gamma_3^2 a_{02}b^2\Delta\left(\Delta\hat{W}_n\right)^2 +$$
$$\left[\lambda_{32}\gamma_2 a_{01}^2 b + \mu_3\gamma_3 a_{11}b + \frac{1}{2}\gamma_3^3 a_{03}b^3 + \gamma_3\left(\lambda_{31}+\lambda_{32}\right)a_{02}ab\right]\Delta^2\left(\Delta\hat{W}_n\right) + \quad (22)$$
$$\left[\lambda_{32}\mu_2 a_{01}a_{10} + \lambda_{32}\gamma_2\gamma_3 a_{01}a_{02}b^2 + \lambda_{21}\lambda_{32}a_{02}^2 a + \mu_3\left(\lambda_{31}+\lambda_{32}\right)\mu_3 a a_{11}\right]\Delta^3 +$$
$$\frac{1}{2}\left[\left(\lambda_{31}+\lambda_{32}\right)^2 a^2 a_{02} + \gamma_3^2\left(\lambda_{31}+\lambda_{32}\right)a_{03}ab^2 + \lambda_{32}\gamma_2^2 a_{01}a_{02}b^2 + \left(a_{20} - \frac{b^4}{4}a_{04}\right)\mu_3^2\right]\Delta^3$$

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Substituting equations (21) and (22) in equation (18), gives the result:

$$\begin{aligned}
&\bar{X}_{n+1} = \bar{X}_n + \beta b\Delta\hat{W}_n + \left(\alpha_1+\alpha_2+\alpha_3\right)a\Delta + \left(\alpha_2\gamma_2+\alpha_3\gamma_3\right)a_{01}b\Delta\left(\Delta\hat{W}_n\right) + \\
&\left[\left(\alpha_2\mu_2+\alpha_3\mu_3\right)a_{10} + \left(\alpha_2\lambda_{21}+\alpha_3\left(\lambda_{31}+\lambda_{32}\right)\right)aa_{10}\right]\Delta^2 + \\
&\left[\alpha_3\lambda_{32}\gamma_2 a_{01}^2 b + \left(\alpha_2\mu_2\gamma_2+\alpha_3\mu_3\gamma_3\right)a_{11}b + \left(\alpha_2\lambda_{21}\gamma_2+\alpha_3\gamma_3\left(\lambda_{31}+\lambda_{32}\right)\right)a_{02}ab\right]\Delta^2\left(\Delta\hat{W}_n\right) + \\
&\frac{1}{2}\left(\alpha_2\gamma_2^3+\alpha_3\gamma_3^3\right)a_{03}b^3\Delta^2\left(\Delta\hat{W}_n\right) + \frac{1}{2}\left(\alpha_2\gamma_2^2+\alpha_3\gamma_3^2\right)a_{02}b^2\Delta\left(\Delta\hat{W}_n\right)^2 + \alpha_2\mu_2\lambda_{32}a_{01}a_{10}\Delta^3 + \\
&\left[\alpha_3\lambda_{21}\lambda_{32}a_{01}^2 a + \frac{1}{2}\alpha_3\lambda_{32}\gamma_2\left(\gamma_2+2\gamma_3\right) + \frac{1}{2}\left(\alpha_2\mu_2^2+\alpha_3\mu_3^2\right)\left(a_{02}-\frac{1}{4}b^4 a_{04}\right)\right]\Delta^3 + \\
&\frac{1}{2}\left[\left(\alpha_2\mu_2\gamma_2^2+\alpha_3\mu_2\gamma_3^2\right)\left(a_{12}b^2+\frac{1}{2}b^4 a_{04}\right) + \left(\alpha_2\lambda_{21}\gamma_2^2+\alpha_3\gamma_3^2\left(\lambda_{31}+\lambda_{32}\right)\right)a_{03}ab^2\right]\Delta^3 + \\
&\left[\left(\alpha_2\lambda_{21}^2+\alpha_3\left(\lambda_{31}+\lambda_{32}\right)^2\right)a_{02}a^2 + \left(\alpha_2\lambda_{21}\mu_2+\alpha_3\mu_3\left(\lambda_{31}+\lambda_{32}\right)\right)a_{11}a\right]\Delta^3 + R
\end{aligned} \tag{23}$$

The previous approximation given in equation (23) is compared with Taylor's simplified third order scheme [13], which corresponds to

$$\begin{aligned}
&\bar{X}_{n+1} = \bar{X}_n + a\Delta + b\Delta\hat{W}_n + a_{01}b\Delta\hat{Z}_n + \frac{1}{2}\left[a_{10}+aa_{01}+\frac{1}{6}b^2 a_{02}\right]\Delta^2 + \\
&\frac{1}{6}\left[ba_{01}^2+2ba_{11}+2baa_{02}+b^3 a_{03}\right]\Delta^2\left(\Delta\hat{W}_n\right) + \frac{1}{6}b^2 a_{02}\Delta\left(\Delta\hat{W}_n\right) + \\
&\frac{1}{6}\left[aa_{01}^2+\frac{3}{2}a_{01}a_{02}b^2+a_{20}+2a_{11}a+a_{12}b^2+a_{03}ab^2+a_{02}a^2+\frac{1}{4}a_{04}b^4+a_{01}a_{10}\right]\Delta^3
\end{aligned} \tag{24}$$

Where $\hat{Z}_n$ is another normal distribution with mean zero and variance 1. In order for the scheme given for equation (23) to be equivalent in the second order to that represented in equation (24) the following equalities must be complied with

$$\begin{array}{lllll}
\beta = 1 & ; & \alpha_2\mu_2^2+\alpha_3\mu_3^2 = \frac{1}{3} & ; & \alpha_2\lambda_{21}+\alpha_3\left(\lambda_{31}+\lambda_{32}\right) = \frac{1}{2} \\
\alpha_1+\alpha_2+\alpha_3 = 1 & ; & \alpha_2\gamma_2^2+\alpha_3\gamma_3^2 = \frac{1}{3} & ; & \alpha_2\gamma_2\lambda_{21}+\alpha_3\gamma_3\left(\lambda_{31}+\lambda_{32}\right) = \frac{1}{3} \\
\alpha_3\gamma_2\lambda_{32} = \frac{1}{6} & : & \alpha_2\gamma_2^3+\alpha_3\gamma_3^3 = \frac{1}{3} & ; & \alpha_2\mu_2\lambda_{21}+\alpha_3\mu_3\left(\lambda_{31}+\lambda_{32}\right) = \frac{1}{3} \\
\alpha_3\mu_2\lambda_{32} = \frac{1}{6} & ; & \alpha_2\mu_2\gamma_2+\alpha_3\mu_3\gamma_3 = \frac{1}{3} & ; & \alpha_2\lambda_{21}\gamma_2^2+\alpha_3\left(\lambda_{31}+\lambda_{32}\right)\gamma_3^2 = \frac{1}{3} \\
\alpha_3\lambda_{21}\lambda_{32} = \frac{1}{6} & ; & \alpha_2\mu_2\gamma_2^2+\alpha_3\mu_3\gamma_3^2 = \frac{1}{3} & : & \alpha_2\lambda_{21}^2+\alpha_3\left(\lambda_{31}+\lambda_{32}\right)^2 = \frac{1}{3} \\
\alpha_2\mu_2+\alpha_3\mu_3 = \frac{1}{2} & ; & \alpha_3\gamma_2\lambda_{32}\left(\gamma_2+2\gamma_3\right) = \frac{1}{2} & &
\end{array} \tag{25}$$

With $R = ba_{01}\left[\Delta\hat{Z}_n - \left(\alpha_2\gamma_2+\alpha_3\gamma_3\right)\Delta\left(\Delta\hat{W}_n\right)\right] + \frac{b^2}{12}a_{02}\Delta^2$.

Simplifying the equivalencies found in equation (25) we obtain,

$$\alpha_2 = \frac{\frac{1}{2}\mu_3 - \frac{1}{3}}{\mu_2\left(\mu_3-\mu_2\right)} \quad ; \quad \alpha_3 = \frac{\frac{1}{3}-\frac{1}{2}\mu_3}{\mu_3\left(\mu_3-\mu_2\right)} \quad ; \quad \begin{array}{l}\mu_2 \neq 0 \\ \mu_3 \neq 0 \\ \mu_3 \neq \mu_2\end{array} \tag{26}$$

The family of parameters that gives a solution to equation (23) is obtained by calculating the polynomial roots $6\mu_3^3 - 17\mu_3^2 + 15\mu_3 - 4 = 0$. The first root $\mu_3 = 1$ is not possible to obtain a family of parameters because it does not comply with the condition $\mu_3 \neq \mu_2$. The family of parameters for the second root $\mu_3 = \frac{1}{2}$ is,

$$\begin{array}{llll}
\lambda_{31} = \frac{13}{32} & \alpha_1 = \frac{1}{12} & \alpha_3 = \frac{8}{9} & \mu_2 = \gamma_2 = 2 \\
\lambda_{32} = \frac{3}{32} & \alpha_2 = \frac{1}{36} & \gamma_3 = \frac{1}{12} & \lambda_{21} = 2
\end{array} \tag{27}$$

Replacing the family of parameters of the second root described in equation (27) in equation (18), gives the result:

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$$\bar{X}_{n+1} = \bar{X}_n + \frac{1}{12}a\Delta + b\Delta\hat{W}_n + \frac{1}{12}b^2 a_{02}\Delta^2 + \frac{1}{36}a\left(t_n + 2\Delta, \bar{X}_n + 2b\Delta\hat{W}_n + 2a\Delta\right)\Delta + \frac{8}{9}a\left(t_n + \frac{\Delta}{2}, \bar{X}_n + \frac{1}{2}b\Delta\hat{W}_n + \frac{13}{32}a\Delta + \frac{3}{32}a\left(t_n + 2\Delta, \bar{X}_n + 2b\Delta\hat{W}_n + 2a\Delta\right)\Delta\right)\Delta + ba_{01}\left[\Delta\hat{Z}_n - \frac{1}{2}\Delta\left(\Delta\hat{W}_n\right)\right] \quad (28)$$

The family of parameters for the third root $\mu_3 = \frac{4}{3}$

$$\lambda_{31} = -\frac{8}{3} \quad \alpha_1 = -\frac{1}{8} \quad \alpha_3 = \frac{1}{3} \quad \mu_2 = \gamma_2 = \frac{1}{3}$$
$$\lambda_{32} = 4 \quad \alpha_2 = 1 \quad \gamma_3 = \frac{4}{3} \quad \lambda_{21} = \frac{1}{3} \quad (29)$$

Substituting the family of parameters for the third root given in equation (29) in equation (18), we obtain:

$$\bar{X}_{n+1} = \bar{X}_n - \frac{1}{8}a\Delta + b\Delta\hat{W}_n + \frac{1}{12}b^2 a_{02}\Delta^2 + a\left(t_n + \frac{1}{3}\Delta, \bar{X}_n + \frac{1}{3}b\Delta\hat{W}_n + \frac{1}{3}a\Delta\right)\Delta + \frac{1}{8}a\left(t_n + \frac{4}{3}\Delta, \bar{X}_n + \frac{4}{3}b\Delta\hat{W}_n - \frac{8}{3}a\Delta + 4a\left(t_n + \frac{1}{3}\Delta, \bar{X}_n + \frac{1}{3}b\Delta\hat{W}_n + \frac{1}{3}a\Delta\right)\Delta\right)\Delta + ba_{01}\left[\Delta\hat{Z}_n - \frac{1}{2}\Delta\left(\Delta\hat{W}_n\right)\right] \quad (30)$$

RESULTS

This section analyzes the precision of the explicit third order, third stage stochastic Runge-Kutta method (RK3-3st) given by equation (28) for the study of point kinetic equations for different forms of reactivity, groups of precursors, initial conditions and physical parameters. Several numerical experiments, implemented with 5000 Brownian trajectories for pseudo-random numbers with normal distribution with seed 200. The calculation of the mean values and standard deviation of the neutron densities and the sum of the precursor densities with the explicit RK3-3st method, are compared with stochastic methods reported in literature, such as: PCA, Monte Carlo, Taylor 1.5, AEM, ESM, Double DDM and EM implicit. For comparison with mean values, the deterministic fourth order Runge-Kutta method (RK $O(h^4)$) is used, with step size $\Delta = 0.001$ s. The proposed numeric experiments are shown, and for these, three densities were considered with different random samples, and average density.

a) Insertion by step reactivity with one precursor: Figure-1 shows the neutron density and the density of precursors according to the first proposed numerical experiment, which considers the following parameters. Decay constant $\lambda_1 = 0.1\ \text{s}^{-1}$, fraction of delayed neutrons $\beta = \beta_1 = 0.005$, average number of fission neutrons $\upsilon = 2.5$, external neutron source $q = 200\ \text{s}^{-1}$, time of neutron generation $\Lambda = \frac{2}{3}\ \text{s}^{-1}$; we used a reactivity constant $\rho = -\frac{1}{3}$, $N = 40$ steps in a time interval $[0,2]$ s and the initial condition assumes equilibrium values $n_{(0)} = 400$ and $c_{1(0)} = 300$.

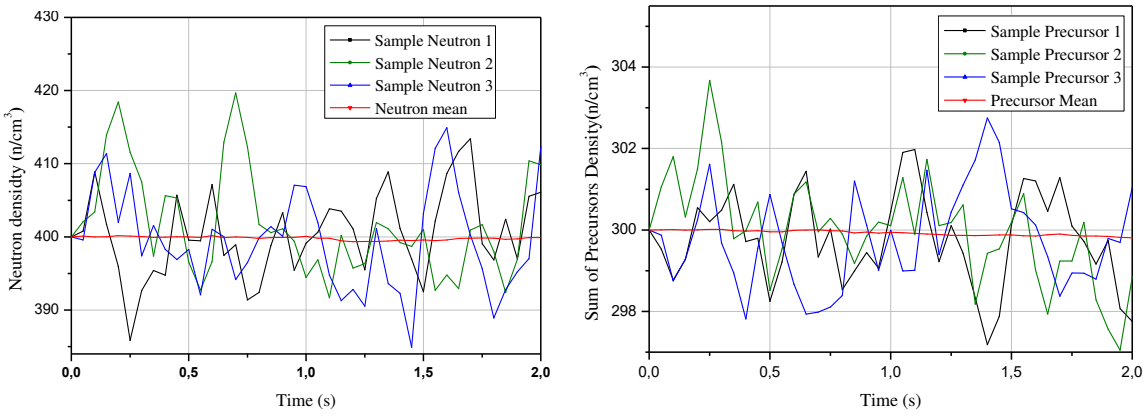

Figure-1. Variation of neutron density and the sum precursor density for the dynamics of a nuclear reactor with step reactivity insertion $\rho = -\frac{1}{3}$.

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Table-2 presents the mean values and standard deviation of neutron density and the concentration of the first group of delayed neutron precursors. The RK3-3st method does not have drawbacks with the problem of rigidity contained in the stochastic point kinetic equations, the method presents approximations very close to the Monte Carlo method in the average values of neutron density and precursor concentration; The variance for the sum of precursors with this method is less than the PCA and EM method, and the variance for the density of neutrons is lower than the EM method and very similar to the Monte Carlo, PCA, AEM and Double DDM methods. On comparing the results obtained with the method proposed for the average values of neutron density, and the sum of neutron density, with the deterministic method RKO, it is found that the proposed method is necessary to calculate the point kinetic equations with step reactivity insertion with one group of precursors.

Table-2. Comparison of methods for step reactivity insertion $\rho = -\frac{1}{3}$ with a group of precursors.

Method	$E\left[n_{(2)}\right]$	$\sigma\left[n_{(2)}\right]$	$E\left[c_{1(2)}\right]$	$\sigma\left[c_{1(2)}\right]$
Monte Carlo	400.0300	27.3110	300.0000	7.8073
PCA	395.3200	29.4110	300.6700	8.3564
EM	412.2300	34.3910	315.9600	8.2656
AEM	396.2800	31.2120	300.4200	7.9576
ESM	396.6200	0.91990	300.3000	0.0016
Double DDM	402.3500	28.6100	305.8400	7.9240
Implicit EM	399.9874	0.5439	299.8730	6.8405
RK3-3st	399.9079	31.8857	299.8090	8.1971
RK O(h^4)	400.0000	-	300.0000	-

b) Insertion by step reactivity with six precursors: In this study the following parameters which are characteristics of $^{235}_{92}U$: decay constant $\lambda_1 = 0.0127\ s^{-1}$; $\lambda_2 = 0.0317\ s^{-1}$; $\lambda_3 = 0.115\ s^{-1}$; $\lambda_4 = 0.311\ s^{-1}$; $\lambda_5 = 1.4\ s^{-1}$ and $\lambda_6 = 3.87\ s^{-1}$, fraction of delayed neutrons $\beta_1 = 0.000266$; $\beta_2 = 0.001491$; $\beta_3 = 0.001316$; $\beta_4 = 0.002849$; $\beta_5 = 0.000896$ and $\beta_6 = 0.000182$, total fraction of delayed neutrons $\beta = 0.007$, fission neutrons $\upsilon = 2.5$, external neutron source $q = 0\ s^{-1}$, time of neutron generation $\Lambda = 0.00002$ s. The initial conditions of the system are $n_{(0)} = 100$ and $c_{i(0)} = 100\,\beta_i/\lambda_i\Lambda$, for $N = 40$ steps. In this example we present two cases of step reactivity insertion: $\rho = 0.003$ in a time interval $[0, 0.1]$ s and $\rho = 0.007$ in $[0, 0.001]$ s. The results of the neutron density and the sum of the precursor densities according to the time for $\rho = 0.003$ are shown in Figure-2 and those for $\rho = 0.007$, can be seen in Figure-3.

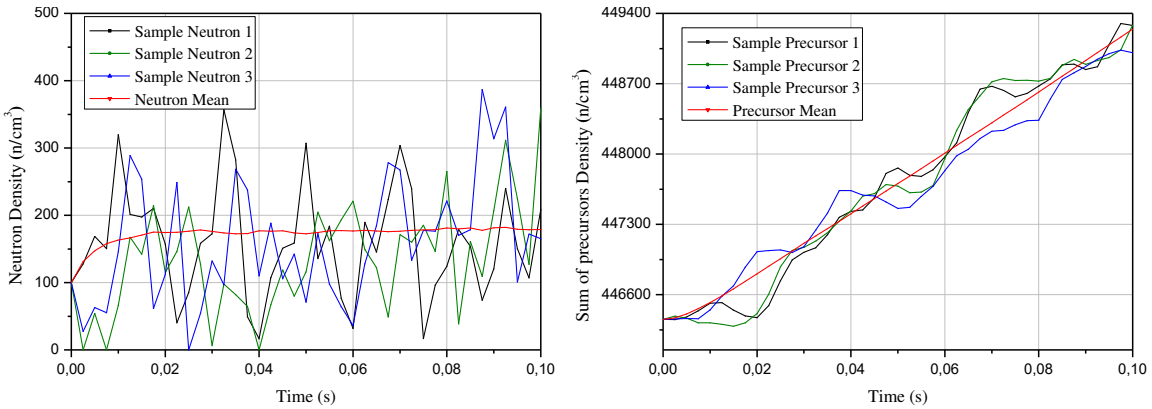

Figure-2. Variation of neutron density and the sum of precursor density for the dynamics of a nuclear reactor with step reactivity insertion $\rho = 0.003$.

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Table-3 presents the average values and standard deviation of neutron density and the sum of the precursor groups obtained for reactivity $\rho = 0.003$. The RK3-3st method obtains average values which are very close to those reported in literature; the standard deviation of neutron density obtained with the RK3-3st method is less than that reported with the EM method, and standard deviation of the sum of precursor density is similar to that reported with the Monte Carlo, PCA, EM, Taylor 1.5 and AEM methods. The mean values and standard deviation of neutron population density, and the sum of precursor density obtained with the RK3-3st method for the second step reactivity insertion case are found in Table-4. It is noted that the average values found are of very good approximation for the density of neutron population, and the sum of density of precursors. The standard deviations calculated by the RK3-3st method are similar to those reported by Monte Carlo, PCA, EM, Taylor 1.5, AEM and Double DDM. The comparison between the average values obtained with the RK3-3st proposed method and the deterministic fourth order Runge-Kutta method can be seen in Table-4. The proposed method is effective in calculating point kinetic equations with constant reactivity and six groups of delayed neutron precursors.

Table-3. Comparison of methods for step reactivity insertion $\rho = 0.003$ with six groups of precursors.

Method	$E\left[n_{(0.1)}\right]$	$\sigma\left[n_{(0.1)}\right]$	$E\left[c_{i(0.1)}\right]$	$\sigma\left[c_{i(0.1)}\right]$
Monte Carlo	183.0400	168.7900	447800	1495.7000
PCA	186.3100	164.1600	449100	1917.2000
EM	208.6000	255.9500	449800	1233.3800
Taylor 1.5	199.4080	168.5470	449700	1218.820
AEM	186.3000	164.1400	449000	1911.9100
ESM	179.9300	10.5550	448900	94.7500
Double DDM	187.0500	167.8300	448800	1475.6000
Implicit EM	179.9461	0.2178	448880	60.4267
RK3-3st	178.9431	186.7942	448854	2009.1295
RK O(h^4)	179.9528	-	448877	-

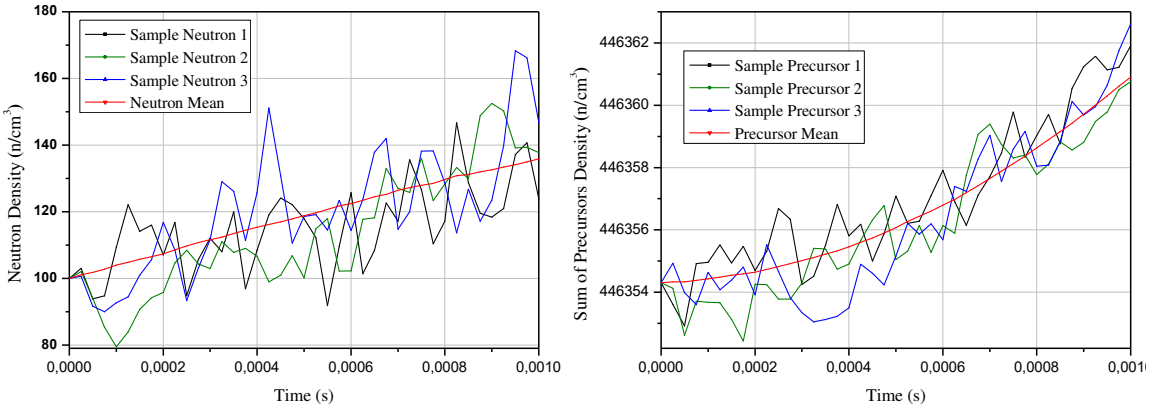

Figure-3. Variation of neutron density and the sum of precursor density for the dynamics of a nuclear reactor with step reactivity insertion $\rho = 0.007$.

Table-4. Comparison of methods for step reactivity insertion $\rho = 0.007$ with six groups of precursors.

Method	$E\left[n_{(0.001)}\right]$	$\sigma\left[n_{(0.001)}\right]$	$E\left[c_{i(0.001)}\right]$	$\sigma\left[c_{i(0.001)}\right]$
Monte Carlo	135.6500	93.3760	446400	16.2260
PCA	134.5500	91.2420	446400	19.4440
EM	139.5680	92.0420	446300	6.0710
Taylor 1.5	139.5690	92.0470	446300	18.3370
AEM	134.5400	91.2340	446400	19.2350

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ESM	134.9600	6.8527	446400	2.5290
Double DDM	135.8600	93.2100	446300	17.8450
Implicit EM	134.9218	5.9661	446360	6.0686
RK3-3st	134.4211	95.4977	446360	19.5491
RK O(h^4)	135.0009	-	446360	-

c) Sinusoidal reactivity: The final experiment presented considers a group of precursors in a nuclear reactor with the following parameters: decay constant $\lambda_1 = 0.077\ s^{-1}$, fraction of delayed neutrons $\beta = \beta_1 = 0.0079$, average number of fission neutrons $\upsilon = 2.5$, external neutron source $q = 0\ s^{-1}$, time of neutron generation $\Lambda = 10^{-3}\ s^{-1}$; a sinusoidal reactivity $\rho_{(t)} = \rho_0 \sin\left(\frac{\pi t}{T}\right)$ was used with an initial reactivity $\rho_0 = 0.005333$ and a period $T = 50\ s$, $N = 100000$ steps in a time interval $[0,100]$ s and an initial condition assumes equilibrium values $n_{(0)} = 1$ and $c_{i(0)} = \beta_i / \lambda_i \Lambda$, with 100 Brownian trajectories. Figure-4 shows the graph of neutron density and the sum of precursor density according to time.

Table-5 presents the mean values and standard deviations of neutron density and the sum of precursor densities. The results obtained for this numerical experiment show that the stochastic RK3-3st method overcomes the Stiffness property of stochastic point kinetics.

The mean deviations of neutron density and the sum of density of precursors found with the RK3-3st method are of greater precision than those reported by the PCA and AEM methods. The standard deviations of neutron density and the sum of precursor density found in the proposed method are less than those reported by the PCA and AEM methods. The explicit RK3-3st method presents mean values compared to the deterministic method of fourth order Runge-Kutta, confirming that the proposed method is efficient for solving the point kinetic equations with sinusoidal reactivity for a group of precursors. But, the results must be improved.

Table-5. Comparison of methods for sinusoidal reactivity $\rho_{(t)} = \rho_0 \sin\left(\frac{\pi t}{T}\right)$ for one group of precursors.

Method	$E\left[n_{(100)}\right]$	$\sigma\left[n_{(100)}\right]$	$E\left[c_{1(100)}\right]$	$\sigma\left[c_{1(100)}\right]$
PCA	187.169345	205.436429	19221.5377	14633.60630
AEM	188.418233	205.776440	19363.9223	14687.71010
ESM	12.539431	2.855346	1294.4320	291.57200
Implicit EM	12.186600	1.550900	1256.6000	159.79770
RK3-3st	23.783789	92.039032	2157.91602	7090.579782
RK O(h^4)	12.439144	-	1282.9033	-

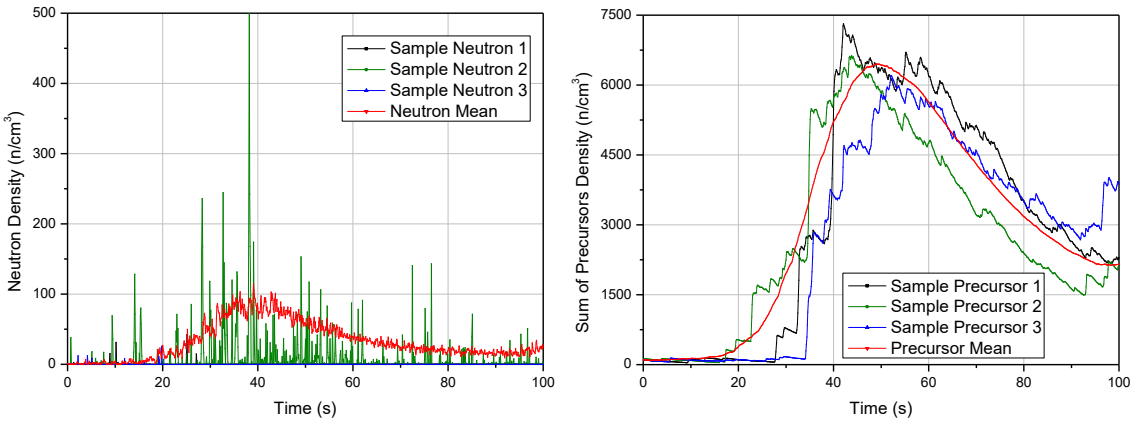

Figure-4. Variation of neutron density and the sum of precursor density for the dynamics of a nuclear reactor with sinusoidal reactivity. $\rho_{(t)} = \rho_0 \sin\left(\frac{\pi t}{T}\right)$.

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CONCLUSIONS

This study employs the RK3-3st method to solve stochastic point kinetic equations. To validate the proposed method, several numerical experiments are performed, in which the insertion of step reactivity with one and six groups of precursors, such as sinusoidal reactivity with a group of precursors, is considered. It is found that the proposed method generates approximations in the average values which are very close to those reported in literature, confirming that in the study of the dynamics of nuclear reactors the explicit method RK3-3st is efficient for solving point kinetic equations with step reactivity insertion. The RK3-3st method has a great advantage over other numerical derivation methods, because it does not require the calculation of any type of derivative for the covariance matrix function.

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