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A NOVEL APPROXIMATION METHOD FOR THE SOLUTION OF CONVECTION-DIFFUSION EQUATION USING BERNSTEIN POLYNOMIALS

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ABSTRACT

In this paper, we have developed operational matrix for estimating the approximate solutions to water quality assessment model. The model is of the form of Convection-Diffusion Equation (CDE) with variable coefficients. An efficient spectral method has been utilized to assess the chemical oxygen demand (COD) in a river. The obtained numerical solutions have been compared with results of Runge-Kutta-Fehlberg fourth-fifth order method (RKF45M) and optimal homotopy asymptotic method (OHAM). The convergence and supporting analysis of the method are investigated Numerical experiments are given to demonstrate the accuracy and efficiency of the proposed method .A few numerical examples are provided to demonstrate the validity and applicability of the proposed method.

Keywords: water quality assessment model, convection-diffusion equation, Bernstein polynomials, operational matrices.

1. INTRODUCTION

The water quality model requires the calculation of the substance dispersion given the water velocity in the channel [1, 2, 12]. Pochai *et al.* [1] developed a finite difference method for water pollution problem. Pochai and his workers [3, 18] had developed a mathematical model of water pollution with the help of finite difference method. Furthermore, Pochai and his coworkers [2-4] had used a few approximation methods for the solution of hydrodynamic model with constant coefficients in the uniform reservoir and stream. The Bernstein polynomial based operational matrix method for assessment of the chemical oxygen demand (COD) in a river is considered. Recently, Padma *et al.* [24] had introduced the homotopy analysis method (HAM) for solving the water quality model in a uniform channel. In this work, the proposed polynomial approximation method has been compared with HAM and finite difference method (FDM). FazleMabood and Pochai [33] introduced a new asymptotic solution for a water quality model.

Recently, Doha *et al.* [26-28] developed a new Chebyshev spectral method for fractional order differential equations. Hariharan and Kannan [25] reviewed the wavelet methods for solving a few reaction-diffusions. Hariharan and his coworkers [6-9] introduced the Haar Wavelet Method (HWM) for solving a few partial differential equations. Ghasemi and Avassoli Kajani [5] applied the Chebyshev wavelet-based method for solving time-varying delay systems. Mason *et al.* [14] used the Chebyshev polynomials for solving differential equations. Hariharan [29] applied the Chebyshev wavelets method (CWM) for solving the convection-diffusion equations. Evans and Abdullah [10] used a new explicit method for the diffusion-convection equation. The boundary element method has been successfully developed for the convection-diffusion equations by Enokizono and Nagata [11]. Rahman and Abuduwaili [13] applied a new numerical method and application for convection diffusion equation.

Horng and Chou [15] used the Chebyshev wavelet spectral method for solving the variational problems. Zhu *et al.* [16, 17] used the CWM for the fractional nonlinear Fredholm integro-differential equations. Sohrabi [19] used the Chebyshev wavelet methods for solving Abel's integral equation. Li [20] presented the Chebyshev wavelet method for fractional order differential equation. Hojatollah Adibi and Pouria Assari [21] used the CWM for solving the Fredholm integral equations. Li Zhu and Qibin Fan [22] established the second kind Chebyshev wavelets for solving fractional nonlinear Fredholm integro-differential equations. Yanxin Wang and Qibin Fan [23] showed the second kind Chebyshev wavelet method for solving fractional differential equations. Yousefi and Behroozifar [30] developed the operational matrix-based polynomial basis for solving differential equations.

2. MATHEMATICAL MODEL [18]

Convection-diffusion equation (CDE) model by means of dispersion of COD is described by [18]

$$-D_x \frac{d^2C}{dx^2} + U\frac{dC}{dx} + RC - Q = 0, \tag{1}$$

where $C(x)$ - Concentration of COD at the point $x \in [a,b]$ (kg/m^3),

U - Flow velocity in x directions (m/s),

D_x - Diffusivity (m^2/s), R - Substrate decay rate (s^{-1}), Q - Increasing rate substrate concentration due to a source $(kg/m^3 s)$.

The corresponding BCs are $C = C_0$ at $x = a$ and $\frac{dC}{dx} = T_0$ at $x = b.$

3. PROPERTIES OF BERNSTEIN POLYNOMIAL [30, 32]

B-Polynomials are defined on the interval [0,1] as

$$b_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}, \mathrm{i} = 0, \ldots, \mathrm{n} \tag{2}$$

$$b_i^n(x) = \sum_{j-i}^{n} (-1)^{j-i} \binom{n}{i}\binom{n-i}{j-i} x^j, i = 0 \ldots\ldots\ldots \mathrm{n} \tag{3}$$

In [24], one can express

$$f(x) \cong \sum_{i=0}^{n} c_i b_i^n = C^T B(x), \tag{4}$$

where the B-coefficient vector C and the Bernstein vector $B(x)$

$$C^T = [c_0, c_1, c_2, \ldots\ldots\ldots\ldots\ldots c_n]$$

$$B(x) = [b_0^n(x), b_1^n(x), \ldots\ldots\ldots\ldots. b_n^n(x)]^T$$

Dual basis functions is defined by the coefficients

$$\lambda_{jk} = \frac{(-1)^{j-k}}{\binom{n}{j}\binom{n}{k}} \sum_{i=0}^{\min(j,k)} (2i+1)\binom{n+i+1}{n-j}\binom{n-i}{n-j}\binom{n+i+1}{n-k}\binom{n-i}{n-k} \tag{5}$$

for $j, k = 0, 1, \ldots, n$

3.1 The operational matrix of the derivative [32]

The operational matrix of derivative of the vector $B(x)$ can be expressed by

$$\frac{dB(x)}{dx} = D^{(1)} B(x), \tag{6}$$

where $D^{(1)}$ is the $(n+1)\times(n+1)$ operational matrix of derivative and is given in [24] as

$$D^{(1)} = AVB^* \tag{7}$$

3.2 Method of solution

Convection-diffusion equation in the form of water quality assessment model [18]

$$c'' = p(x)c' + q(x)c + r(x). \tag{8}$$

Using the aforesaid method described in Section 3 with $m = 2$,

$$C^T D^2 \Psi(x) = p(x) C^T D \Psi(x) + q(x) C^T \Psi(x) + r(x) \tag{9}$$

Then

$$D^{(2)} = \begin{pmatrix} 2 & 2 & 2 \\ -4 & -4 & -4 \\ 2 & 2 & 2 \end{pmatrix}, \; D^{(1)} = \begin{pmatrix} -2 & -1 & 0 \\ -2 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\Psi(x) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} (1-x)^2 \\ 2x(1-x) \\ x^2 \end{pmatrix},$$

$$C^T = \sqrt{\frac{\pi}{2}} \begin{bmatrix} c_0 & c_1 & c_2 \end{bmatrix}$$

4. NUMERICAL EXAMPLES

Example 4.1: Consider the convection-diffusion equation with variable coefficients described in Ref. [18]

$$c'' = p(x)c' + q(x)c + r(x). \tag{10}$$

Let the physical parameter values are: diffusion coefficient 2 $m^2/s,$ flow velocity $u = 5 - x \;\; m/s, \;\; x \in [0, 2],$ substance decay rate $3\, s^{-1}$ and rate of change of substance concentration due to the source 1 $Kg/m^3 s.$

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$$\left.\begin{aligned} p(x) &= \frac{5-x}{D_x} = \frac{5-x}{12} \\ q(x) &= \left(\frac{R}{D_x}\right) = \frac{3}{12} \\ r(x) &= \left(\frac{-Q}{D_x}\right) = \frac{-1}{12}. \end{aligned}\right\} \quad (11)$$

Eq. (10) becomes

$$c'' = \left(\frac{5-x}{12}\right)c' + \left(\frac{3}{12}\right)c - \left(\frac{1}{12}\right). \quad (12)$$

Subject to the conditions $c(0)=1.25$ and $c'(2)=0.$ (13)

We solve Eq. (12) using the procedure described in Section 3, for the case corresponds to $m=2$ to get an approximation solution $c(x)$. B-polynomial operational matrix scheme is given by

$$C^T D^2\psi(x) - \left(\frac{5-x}{12}\right)C^T D\psi(x) - \frac{3}{12}C^T\psi(x) + \frac{1}{12} = 0 \quad (14)$$

$$\frac{31}{8}c_0 + 5c_1 - \frac{5c_2}{8} + \frac{1}{2} = 0 \quad (15)$$

Furthermore, the initial conditions are given by

$$c^T\psi(0) = 1.25 \;\Rightarrow c_0 = 1.25 \quad (16)$$

$$c^T D\psi(2) = 0 \;\Rightarrow c_1 = 0.25 + 0.4c_2 \quad (17)$$

Solving equations (15), (16) and (17), we obtain

$$c_0 = 1.25;\quad c_1 = 0.8479;\quad c_2 = 0.646875$$

Consequently,

$$c(x) = [1.25 \quad 0.8479 \quad 0.646875]\begin{pmatrix}(1-x)^2 \\ 2x(1-x) \\ x^2\end{pmatrix} \quad (18)$$

$$= 0.2011x^2 - 0.8042x + 1.25$$

which is the exact solution.
Zero order problems [33] is

$$c_0'' - \frac{5}{2}c_0' - \frac{3}{2}c_0 + \frac{1}{2} = 0 \quad (19)$$

with the boundary conditions

$$c_0(0) = 1.25,\; c_0'(2) = 0 \quad (20)$$

The solution of (19) with boundary conditions (20) is

$$c_0(x) = \frac{77e^{3x}}{12(6e^7+1)} + \frac{1}{3}. \quad (21)$$

Using B-polynomial operational algorithm, Eq. (19) can be written

$$c^T D^2\psi(x) - \frac{5}{2}c^T D\psi(x) - \frac{3}{2}c^T\psi(x) + \frac{1}{2} = 0 \quad (22)$$

which is equivalent to the following expression

$$\frac{33}{8}c_0 + 6c_1 + \frac{3}{2} = 0 \quad (23)$$

Furthermore, the initial conditions are given by

$$\left.\begin{aligned} c^T\psi(0) = 1.25 &\Rightarrow c_0 = 1.25 \\ c^T D\psi(2) = 0 &\Rightarrow 2c_0 - 10c_1 + 4c_2 = 0 \end{aligned}\right\} \quad (24)$$

Solving Eqs. (23) and (24), we get the wavelet coefficients.

The obtained results have been compared with Pochai [18], Padma *et al.* [24], Fazle Mabood and Pochai's [33] results. By the approximate solutions, it can be obtained that the COD concentration along a uniform channel will be decreasing. Table-1 shows the comparison of RKF45M, optimal homotopy asymptotic method (OHAM) and Bernstein operational matrix method (BOMM).

Table-1. Comparison of $c(x)$ via BOM and other methods.

x	RKF45 [33]	OHAM [33]	Our method BOM
0.0	1.2500	1.2500	1.2500
0.1	1.2031	1.2031	1.2031
0.2	1.1580	1.1580	1.1580
0.3	1.1147	1.1146	1.1145
0.4	1.0731	1.0731	1.0730
0.5	1.0333	1.0332	1.0333
0.6	0.9952	0.9951	0.9952
0.7	0.9588	0.9588	0.9588
0.8	0.9242	0.9242	0.9242
0.9	0.8914	0.8913	0.8912
1.0	0.8604	0.8602	0.8604

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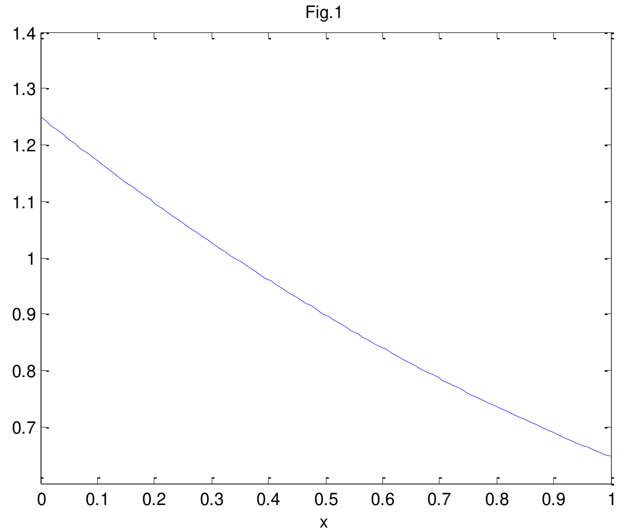

Figure-1. BOM solution for Eq. (18).

Example 4.2: Let the steady-state convection-diffusion equation [9]

$$-\alpha c_{xx} + \beta c_x = 0, \quad c(0) = 0,\ c(1) = 1 \tag{25}$$

where α and β are constants.
Exact solution:

$$c(x) = \frac{e^{\frac{\beta}{\alpha}x} - 1}{e^{\frac{\beta}{\alpha}} - 1} \tag{26}$$

$$\frac{dc}{dx} = \frac{Pe * e^{Pe\,x}}{1 - e^{Pe}} \tag{27}$$

The obtained solution can be compared with the Chebyshev wavelet solution. One increases the p´eclet number Pe from 1 to 10, 50,100 and compares the $\frac{dc}{dx} = \frac{Pe * e^{Pe\,x}}{1 - e^{Pe}}$ at x=1. On the other hand, the Bernstein operational matrix (BOM) method provides a stable and acceptably accurate solution.

Using the aforesaid scheme, one can obtain

$$-\alpha C^T D^2 \psi(x) + \beta C^T D \psi(x) = 0 \tag{28}$$

Setting $\alpha = \beta = 1$, *we get*

$$-c^T D^2 \psi(x) + c^T D \psi(x) = 0 \tag{29}$$

which is equivalent to

$$-3c_0 - c_2 = 0 \tag{30}$$

Furthermore, the initial conditions are given by

$$c^T \psi(0) = 0 \ \Rightarrow c_0 = 0 \tag{31}$$

$$c^T \psi(1) = 1 \quad \Rightarrow \ c_1 + c_2 = 1 \tag{32}$$

Example. 4.3 We consider the convection-diffusion problem [29]

$$-2\left(2x^2 + \tan\left(x^2\right)\right)u = \frac{d^2u}{dx^2} \tag{33}$$

Using the aforesaid scheme, one can obtain

$$-2\left(2x^2 + \tan\left(x^2\right)\right)C^T \psi(x) = C^T D^2 \psi(x) \tag{34}$$

$$c^T D^2 \psi(x) + \left(4x^2 + 2\tan\left(x^2\right)\right)c^T \psi(x) = 0 \tag{35}$$

Using Eq. (35), the linear system can be solved with the aid of Newton's iterative method. It is worth noting that applying the scheme proposed above for the Eq. (35), the solution in a closed form $u(x) = \cos\left(x^2\right),\ x = \left(0, \frac{\pi}{4}\right)$ can be compared with the CW solution. Our results can be compared with Hariharan's results [29]. Figure-1 and Figure-2 show the BOM solutions for various values of x.

Example 4.4 Consider the equation

$$xy''(x) + y'(x) + xy(x) = 0, \tag{36}$$

with the initial conditions

$$y(0) = 1, \mathrm{y}'(0) = 0 \tag{37}$$

Using the aforesaid method with $m = 2$, we approximate solution as Eq. (36) becomes

$$xC^T D^2 \Psi(x) + C^T D \Psi(x) + xC^T \Psi(x) = 0 \tag{38}$$

By collocation Eq. (82) at $x = \frac{1}{2}$, we gain

$$x^3\left(c_0-c_1+c_2\right)+2x^2\left(-c_0+c_1\right)+2x\left(2c_0-4c_1+2c_2\right)+xc_0+2\left(-c_0+c_1\right)=0 \quad (39)$$

Using Eq. (36), we gain

$$c_0-13c_1+17c_2=0, \quad (40)$$

From Eq.(37)

$$y(0)=0\Rightarrow c_0=1$$
$$y'(0)=0\Rightarrow -2c_0+2c_1=0\Rightarrow c_1=1$$

Finally, we gain

$$c_0=1, c_1=1, c_2=\frac{12}{17}$$

Consequently

$$y(x)=\left(1,1,\frac{12}{17}\right)\begin{pmatrix}(1-x)^2\\ 2x(1-x)\\ x^2\end{pmatrix}$$

$$y(x)=1-\frac{5}{17}x^2 \quad (41)$$

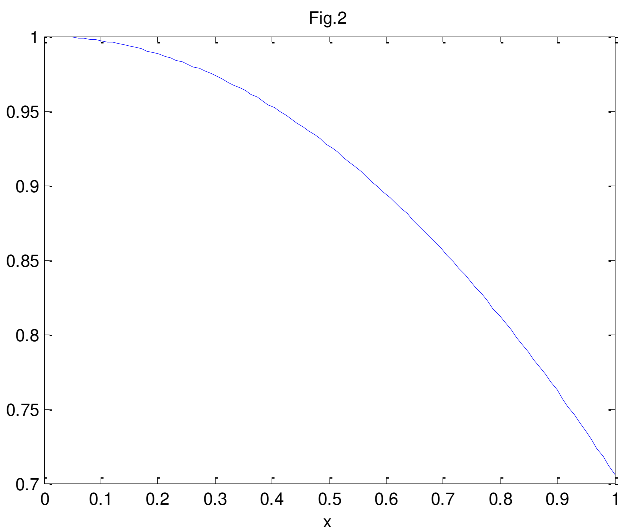

Figure-2. BOM solution for Eq. (41).

5. CONCLUSIONS

An efficient B-Polynomial approximation method has been successfully employed to water quality assessment model problem. Numerical experiments show that the proposed spectral method can match exactly with the analytical solution very efficiently. It has been concluded that the proposed B-polynomial method is very powerful and efficient in finding analytical as well as numerical solutions for nonlinear differential equations.

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