A NOVEL APPROXIMATION METHOD FOR THE SOLUTION OF CONVECTION-DIFFUSION EQUATION USING BERNSTEIN POLYNOMIALS

R. Seethlakshmi, M. Mahalakshmi and G. Hariharan,
Department of Mathematics, School of Arts, Science and Humanities, SASTRA Deemed University, Thanjavur, Tamil Nadu, India
E-Mail: mahalakshmi@maths.sastra.edu

ABSTRACT

In this paper, we have developed operational matrix for estimating the approximate solutions to water quality assessment model. The model is of the form of Convection-Diffusion Equation (CDE) with variable coefficients. An efficient spectral method has been utilized to assess the chemical oxygen demand (COD) in a river. The obtained numerical solutions have been compared with results of Runge-Kutta-Fehlberg fourth-fifth order method (RKF45M) and optimal homotopy asymptotic method (OHAM). The convergence and supporting analysis of the method are investigated. Numerical experiments are given to demonstrate the accuracy and efficiency of the proposed method. A few numerical examples are provided to demonstrate the validity and applicability of the proposed method.

Keywords: water quality assessment model, convection-diffusion equation, Bernstein polynomials, operational matrices.

1. INTRODUCTION

The water quality model requires the calculation of the substance dispersion given the water velocity in the channel [1, 2, 12]. Pochai et al. [1] developed a finite difference method for water pollution problem. Pochai and his workers [3, 18] had developed a mathematical model of water pollution with the help of finite difference method. Furthermore, Pochai and his coworkers [2-4] had used a few approximation methods for the solution of hydrodynamic model with constant coefficients in the uniform reservoir and stream. The Bernstein polynomial based operational matrix method for assessment of the chemical oxygen demand (COD) in a river is considered. Recently, Padma et al. [24] had introduced the homotopy analysis method (HAM) for solving the water quality model in a uniform channel. In this work, the proposed polynomial approximation method has been compared with HAM and finite difference method (FDM). FazleMabood and Pochai [33] introduced a new asymptotic solution for a water quality model.


2. MATHEMATICAL MODEL [18]

Convection-diffusion equation (CDE) model by means of dispersion of COD is described by [18]

\[-D_x \frac{d^2 C}{dx^2} + U \frac{dC}{dx} + RC - Q = 0,\]

where \( C(x) \) - Concentration of COD at the point \( x \in [a, b] \) (kg/m³),

\( U \) - Flow velocity in \( x \) directions (m/s),

\( D_x \) - Diffusivity (m²/s), \( R \) - Substrate decay rate (s⁻¹), \( Q \) - Increasing rate substrate concentration due to a source (kg/m³ s⁻¹).

3726
The corresponding BCs are \( C = C_0 \) at \( x = a \) and \( \frac{dC}{dx} = T_0 \) at \( x = b \).

3. PROPERTIES OF BERNSTEIN POLYNOMIAL [30, 32]

B-Polynomials are defined on the interval \([0,1]\) as

\[
b^n_i(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, \ldots, n
\]

(2)

\[
b^n_i(x) = \sum_{j=1}^{n} (-1)^{j-i} \binom{n}{j} \binom{n-i}{j-i} x^j, \quad i = 0, \ldots, n
\]

(3)

In [24], one can express

\[
f(x) \prod_{i=0}^{n} c b^n_i = C^T B(x),
\]

(4)

where the B-coefficient vector \( C \) and the Bernstein vector \( B(x) \)

\[
C^T = \left[ c_0, c_1, c_2, \ldots, c_n \right]
\]

\[
B(x) = [b^n_0(x), b^n_1(x), \ldots, b^n_n(x)]^T
\]

Dual basis functions is defined by the coefficients

\[
\lambda_{jk} = \left( \frac{-1}{n} \right)^{j-k} \binom{n}{j} \binom{n}{k} \sum_{i=0}^{\min(j,k)} (2i+1) \binom{n+i+1}{n-j} \binom{n-i}{n-j} \binom{n+i+1}{n-k} \binom{n-i}{n-k}
\]

(5)

3.1 The operational matrix of the derivative [32]

The operational matrix of derivative of the vector \( B(x) \) can be expressed by

\[
\frac{dB(x)}{dx} = D^{(1)} B(x),
\]

(6)

where \( D^{(1)} \) is the \((n+1)\times(n+1)\) operational matrix of derivative and is given in [24] as

\[
D^{(1)} = A V B^*
\]

(7)

3.2 Method of solution

Convection-diffusion equation in the form of water quality assessment model [18]

\[
c'' = p(x)c' + q(x)c + r(x).
\]

(8)

Using the aforesaid method described in Section 3 with \( m = 2 \),

\[
C^T D^2 \Psi(x) = p(x)C^T D\Psi(x) + q(x)C^T \Psi(x) + r(x)
\]

(9)

4. NUMERICAL EXAMPLES

Example 4.1: Consider the convection-diffusion equation with variable coefficients described in Ref. [18]

\[
c'' = p(x)c' + q(x)c + r(x).
\]

(10)

Let the physical parameter values are: diffusion coefficient \( 2 m^2/s \), flow velocity \( u = 5 - x \) \( m/s \), \( x \in [0,2] \), substance decay rate \( 3 s^{-1} \) and rate of change of substance concentration due to the source \( 1 Kg/m^3 s \).
The solution of (19) with boundary conditions (20) is

\[ c_0(x) = -\frac{77e^{-3x}}{12(6e^7+1)} + \frac{1}{3}. \] (21)

Using B-polynomial operational algorithm, Eq. (19) can be written

\[ c^T D^2 \psi(x) - \frac{5}{2} c^T D \psi(x) - \frac{3}{2} c^T \psi(x) + \frac{1}{2} = 0 \] (22)

which is equivalent to the following expression

\[ \frac{33}{8} c_0 + 6c_1 + \frac{3}{2} = 0 \] (23)

Furthermore, the initial conditions are given by

\[ c^T \psi(0) = 1.25 \Rightarrow c_0 = 1.25 \] (16)

\[ c^T D \psi(2) = 0 \Rightarrow c_1 = 0.25 + 0.4c_2 \] (17)

Solving equations (15), (16) and (17), we obtain

\[ c_0 = 1.25; \quad c_1 = 0.8479; \quad c_2 = 0.646875 \]

Consequently,

\[ c(x) = [1.25 \quad 0.8479 \quad 0.646875] \begin{bmatrix} (1-x)^2 \\ 2x(1-x) \\ \frac{ax^3}{3} \end{bmatrix} \] (18)

which is the exact solution.

Zero order problems [33] is

\[ c_0^0 \frac{5}{2} c_0' - \frac{3}{2} c_0 + \frac{1}{2} = 0 \] (19)

with the boundary conditions

\[ c_0(0) = 1.25, \quad c_0'(2) = 0 \] (20)

Table 1. Comparison of \( c(x) \) via BOM and other methods.

<table>
<thead>
<tr>
<th>x</th>
<th>RKF45 [33]</th>
<th>OHAM [33]</th>
<th>Our method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.2500</td>
<td>1.2500</td>
<td>1.2500</td>
</tr>
<tr>
<td>0.1</td>
<td>1.2031</td>
<td>1.2031</td>
<td>1.2031</td>
</tr>
<tr>
<td>0.2</td>
<td>1.1580</td>
<td>1.1580</td>
<td>1.1580</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1147</td>
<td>1.1146</td>
<td>1.1145</td>
</tr>
<tr>
<td>0.4</td>
<td>1.0731</td>
<td>1.0731</td>
<td>1.0730</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0333</td>
<td>1.0332</td>
<td>1.0333</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9952</td>
<td>0.9951</td>
<td>0.9952</td>
</tr>
<tr>
<td>0.7</td>
<td>0.9588</td>
<td>0.9588</td>
<td>0.9588</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9242</td>
<td>0.9242</td>
<td>0.9242</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8914</td>
<td>0.8913</td>
<td>0.8912</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8604</td>
<td>0.8602</td>
<td>0.8604</td>
</tr>
</tbody>
</table>
Example 4.2: Let the steady-state convection-diffusion equation [9]

\[-\alpha c_{xx} + \beta c_x = 0, \quad c(0) = 0, \quad c(1) = 1 \quad (25)\]

where \( \alpha \) and \( \beta \) are constants.

Exact solution:

\[c(x) = \frac{\beta x}{e^{\alpha x}} - 1 \quad (26)\]

\[\frac{dc}{dx} = \frac{Pe^* e^{Pe x}}{1 - e^{Pe x}} \quad (27)\]

The obtained solution can be compared with the Chebyshev wavelet solution. One increases the Péclet number \( Pe \) from 1 to 10, 50, 100 and compares the Bernstein operational matrix (BOM) method provides a stable and acceptably accurate solution.

Using the aforesaid scheme, one can obtain

\[-3c_0 - c_2 = 0 \quad (30)\]

Furthermore, the initial conditions are given by

\[c^T \psi(0) = 0 \Rightarrow c_0 = 0 \quad (31)\]

\[c^T \psi(1) = 1 \Rightarrow c_1 + c_2 = 1 \quad (32)\]

Example 4.3 We consider the convection-diffusion problem [29]

\[-2\left(2x^2 + \tan(x^2)\right)u = \frac{d^2u}{dx^2} \quad (33)\]

Using the aforesaid scheme, one can obtain

\[-2\left(2x^2 + \tan(x^2)\right)C^T \psi(x) = C^T D^2 \psi(x) \quad (34)\]

\[c^T D^2 \psi(x) + \left(4x^2 + 2\tan(x^2)\right)c^T \psi(x) = 0 \quad (35)\]

Using Eq. (35), the linear system can be solved with the aid of Newton’s iterative method. It is worth noting that applying the scheme proposed above for the Eq. (35), the solution in a closed form can be compared with the CW solution. Our results can be compared with Hariharan’s results [29]. Figure-1 and Figure-2 show the BOM solutions for various values of \( x \).

Example 4.4 Consider the equation

\[x y''(x) + y'(x) + xy(x) = 0 \quad (36)\]

with the initial conditions

\[y(0) = 1, \quad y'(0) = 0 \quad (37)\]

Using the aforesaid method with \( m = 2 \), we approximate solution as Eq. (36) becomes

\[x C^T D^2 \Psi(x) + C^T D \Psi(x) + x C^T \Psi(x) = 0 \quad (38)\]

By collocation Eq. (82) at \( x = \frac{1}{2} \), we gain

\[-3c_0 - c_2 = 0 \quad (30)\]
\[ x^3(c_0 - c_1 + c_2) + 2x^2(-c_0 + c_1) + 2x(2c_0 - 4c_1 + 2c_2) + xc_0 + 2(-c_0 + c_1) = 0 \]  

(39)

Using Eq. (36), we gain

\[ c_0 - 13c_1 + 17c_2 = 0. \]  

(40)

From Eq.(37)

\[ y(0) = 0 \Rightarrow c_0 = 1 \]

\[ y'(0) = 0 \Rightarrow -2c_0 + 2c_1 = 0 \Rightarrow c_1 = 1 \]

Finally, we gain

\[ c_0 = 1, c_1 = 1, c_2 = \frac{12}{17} \]

Consequently

\[ y(x) = \left(1, \frac{12}{17}\right) \begin{pmatrix} (1-x)^2 \\ 2x(1-x) \\ x^2 \end{pmatrix} \]

\[ y(x) = 1 - \frac{5}{17} x^3 \]  

(41)

5. CONCLUSIONS

An efficient B-Polynomial approximation method has been successfully employed to water quality assessment model problem. Numerical experiments show that the proposed spectral method can match exactly with the analytical solution very efficiently. It has been concluded that the proposed B-polynomial method is very powerful and efficient in finding analytical as well as numerical solutions for nonlinear differential equations.

REFERENCES


