FINITE FOURIER SINE INTEGRAL TRANSFORM METHOD FOR THE ELASTIC BUCKLING ANALYSIS OF DOUBLY-SYMMETRIC THIN-WALLED BEAMS WITH DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT

The finite Fourier sine transform method was used in this work to solve the elastic buckling problem of thin-walled beams for the case of pinned ends, and uniform moments applied at the ends. The problem is a boundary value problem given by a fourth order ordinary differential equation and Dirichlet boundary conditions at the pinned ends. The Dirichlet boundary conditions at the pinned ends make the finite Fourier sine transform method ideally suited for the solution. The transformation of the governing domain equation converted the problem to an algebraic eigenvalue problem. The condition for nontrivial solution was used to obtain the characteristic buckling equation as a fourth degree polynomial. The eigenvalues of the characteristic buckling equation were used to obtain the n buckling moments. The critical buckling moment was found to correspond to the first buckling mode. The expressions obtained for the n buckling modes and the critical buckling moment were identical to those by other researchers who used other methods of analysis.

Keywords: Finite Fourier sine transform method, algebraic eigenvalue problem, characteristic buckling equation, Dirichlet boundary conditions, critical buckling moment, thin-walled beams with doubly-symmetric cross-sections.

INTRODUCTION

Thin-walled steel beam structures are commonly used in structural applications due to the high strength of steel. The small thicknesses of the cross-section walls make them susceptible to instabilities [1 - 9]. One of the general forms of instabilities that could develop for thin-walled beams in bending is the lateral - torsional buckling. Consequently, in the analysis and design of thin-walled steel beams, lateral torsional buckling analysis should be considered as it can significantly reduce the load carrying capacity, and thus affect the safety of the entire structures [7 - 11].

At the imminence of lateral - torsional buckling, the structural behaviour charges from mainly in-plane bending to combined lateral deflection and twisting, and it is one of the most significant stability problems, which may frequently be a controlling consideration in the analysis and design of steel beams [12, 13, 14].

Lateral torsional buckling (LTB) is the buckling of a thin-walled beam loaded in the plane of its strong axis and buckling about the weaker axis accompanied by twisting (torsion). The load at which such a beam buckles can be much less than the load causing the development of the full moment capacity of the beam. LTB should be investigated in slender (thin walled) beams that have greater major axis bending stiffness then minor axis bending stiffness, or considerably large laterally unsupported lengths.

Lateral torsion - flexure buckling of thin-walled steel I - beams subject to flexure is hence a vital consideration in their analysis and design [15 - 21].

In general, when a slender steel beam with I – cross section is subjected to flexural loads about its axis of greatest flexural rigidity and the beam has insufficient lateral bracing, out - of - plane bending and twisting may occur when the applied load attains a certain critical value. At this critical load, the beam is said to have failed by lateral torsion flexure [15, 22 - 25].

Thus, lateral - torsional buckling (LTB) phenomenon occurs when the bending action attains a certain critical load. It generates a sudden and simultaneous lateral bending deformation and a longitudinal torsional deformation along the unrestrained length of the beam.

The critical load that causes lateral buckling has been found in the literature to depend upon the laterally unbraced length of the beam, the type of loading, the location of the load with respect to the shear centre of the beam cross-section, the mechanical properties of the beams cross-section; geometric properties of the cross-section such as the torsion constant, warping constant and the moment of inertia about the minor axis, material properties such as the Young's modulus of elasticity and the shear modulus [26 – 31]. Lateral torsional buckling failure can occur in straight single or multi-span beams, under simple bending about their strong axis, with bi-symmetric or mono-symmetric cross-sections about their plane of bending.

The position of the applied load with respect to the shear centre of the beam cross-section is often neglected by most codes of practice for steel design despite that loads can be applied at an eccentricity with respect to the shear centre of the cross-section.

For design purposes, most codes of practice permit the application of equivalent moment factors as modification factors to the critical moment causing lateral

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torsion flexure buckling for simply supported beam subjected to uniform bending moment in order to determine the critical buckling load capacities for other load types.

Lateral torsional buckling failures can occur in doubly symmetric I shaped (cross-section) beams, I section loaded in the plane of their webs, I section singly symmetric with compression flanges. It is not possible for LTB to occur if the moment of inertia of the cross-sections about the bending axis is equal to or less than moment of inertia out of plane. Hence, the limit state of LTB is not applicable for shapes bent about their minor axis, and for circular or square cross sections.

A beam can fail by reaching the plastic moment and becoming fully plastic or fail prematurely by lateral torsional buckling (LTB) (elastically or inelastically) flange local buckling (FLB) (elastically or inelastically) or web local buckling (WLB) (elastically or inelastically) [4].

If the maximum bending stress is less than the proportional limit of elasticity at the incipience of buckling failure, the buckling failure is called elastic; otherwise, the buckling failure is called inelastic.

Attard and Kim [12] derived the equilibrium and buckling equations for the lateral buckling of a straight beam with prismatic cross-section considering shear deformations and using hyperelastic formulation. They used a consistent finite strain constitutive law, based on a hyperelastic model for an isotropic material. They also derived second order approximations to the displacements, curvatures, twist and internal actions. They found the constitutive relations for the internal actions to contain new coupling terms between the bending moments, torsion and bimoment which depend on the cross-sectional warping and shear deformations. They obtained solutions to the lateral buckling problems of prismatic, monosymmetric beams under pure flexure and the flexural torsional buckling under axial compression.

Chan *et al* [24] presented a novel approach for the determination of the critical lateral - torsional buckling loads of beams subjected to arbitrary transverse loads. Their new formulation was based on classical energy methods. The difference between their method and the traditional energy methods is the formulation of the potential energy of the external loads which was expressed in terms of internal bending moment and internal shear force in the pre-buckling stage regardless of the type of loading. Their presentation was simple and unified for the accurate calculation of critical load of lateral torsional buckling of beams.

Bijak [32, 33] obtained approximate solutions to the stability equations for simply supported, unrestrained, monosymmetric beams with the influence of displacements in the plane of bending using the Bubnov - Galerkin method. He found that the lateral - torsional buckling (LTB) moment depends on the bending distribution and on the load height effect. He solved numerical problems to show good consistency of his obtained results with those obtained using Finite Element Method (FEM) software.

Piotrowski and Szychowski [34] presented theoretical investigations into the lateral torsional buckling of bisymmetric I-section beams elastically restrained against warping and rotation in the plane of lateral torsional buckling (i.e. against lateral rotation) at the support nodes. Their presented analysis considered the full range of variation of node stiffnesses from complete warping freedom to full warping restraint, and from complete lateral rotation freedom to full lateral rotation restraint. They further assumed the beams to be simply supported against bending about the major axis of the cross-section. They determined the critical lateral - torsional buckling moment by using polynomials to describe the twist angle function and the lateral deflection function of the beam and the aid of computer programmes based on symbolic algebra computations; and obtained satisfactory results when compared with results from FEM software.

In this work, the finite Fourier sine integral transform method is used as an analytical tool to solve the boundary value problem (BVP) of the elastic buckling analysis of thin-walled beams (under uniform bending moments at the supports) with doubly symmetric cross-sections, for the case of simply supported ends.

THEORETICAL FRAMEWORK

The elastic buckling analysis of simply supported thin-walled beams with doubly-symmetric cross-sections is considered. The beam cross-section is in the *xy* coordinate plane while the *z* axis is the longitudinal axis of the beam of length, *l*. The case of uniaxial bending for the specific case of a uniform moment M_0 causing compression in the top flange, the governing equation for the elastic buckling of the beam is given by the fourth order ordinary differential equation [10]:

$$EI_w \phi^{iv}(z) - GJ\phi''(z) - \frac{M_0^2}{EI_y}\phi(z) = 0 \qquad (1)$$

(for $0 \le z \le l$)

where

E is the Young's modulus of elasticity

I_w is the warping constant or Saint Venant warping constant

EI_y is the flexural rigidity (minor axis)

G is the shear modulus, or the modulus of rigidity

J is the Saint Venant torsion constant

I_y is the moment of inertia

$\phi(z)$ is the twist (rotational) displacement

The primes denote differentiation with respect to the longitudinal axis coordinate, *zz* is the longitudinal coordinate axis, and *l* is the unbraced length of the beam subjected to constant moment in plane of the web.

Equation (1) is the domain governing differential equation for the lateral torsional buckling (LTB) of beams with symmetric cross-sections. Equation (1) can be expressed alternatively as:

$$\phi^{iv}(z) - \frac{GJ}{EI_w}\phi''(z) - \frac{M_0^2\phi(z)}{EI_w EI_y} = 0 \quad (2)$$

or,

$$\phi^{iv}(z) - \alpha_1\phi''(z) - \alpha_2\phi(z) = 0 \quad (3)$$

where, $\alpha_1 = \dfrac{GJ}{EI_w}$ (4)

$$\alpha_2 = \frac{M_0^2}{EI_w EI_y} \quad (5)$$

METHOD

Finite Fourier sine transformation method

For thin-walled beams with simply supported ends, the Dirichlet boundary conditions which are satisfied by the kernel function of the finite Fourier sine transform are given by Equations (6) and (7).

$$\phi(z=0) = \phi(z=l) = 0 \quad (6)$$

$$\phi''(z=0) = \phi''(z=l) = 0 \quad (7)$$

Hence, Equation (3) can be solved by the finite Fourier sine transformation method. Application of the finite Fourier sine transformation to Equation (3) yields:

$$\int_0^l (\phi^{iv}(z) - \alpha_1\phi''(z) - \alpha_2\phi(z))\sin\frac{n\pi z}{l}\,dz = 0 \quad (8)$$

$n = 1, 2, 3, 4, \ldots$

or, $\displaystyle\int_0^l \left(\frac{d^4\phi(z)}{dz^4} - \alpha_1\frac{d^2\phi(z)}{dz^2} - \alpha_2\phi(z)\right)\sin\frac{n\pi z}{l}\,dz = 0$ (9)

Application of the linearity properties of the finite Fourier sine transform, and simplification yields:

$$\left(\frac{n\pi}{l}\right)^4\int_0^l \phi(z)\sin\frac{n\pi z}{l}dz + \left(\frac{n\pi}{l}\right)^2\alpha_1\int_0^l \phi(z)\sin\frac{n\pi z}{l}dz - \alpha_2\int_0^l \phi(z)\sin\frac{n\pi z}{l}dz = 0 \quad (10)$$

Hence we denote:

$$\int_0^l \phi(z)\sin\frac{n\pi z}{l}\,dz = \Omega(n) \quad (11)$$

where $\Omega(n)$ is the finite Fourier sine transform of $\phi(z)$.
Equation (10) becomes the algebraic equation:

$$\left(\frac{n\pi}{l}\right)^4\Omega(n) + \left(\frac{n\pi}{l}\right)^2\alpha_1\Omega(n) - \alpha_2\Omega(n) = 0 \quad (12)$$

Factoring out $\Omega(n)$ we have the algebraic eigenvalue problem:

$$\left(\left(\frac{n\pi}{l}\right)^4 + \left(\frac{n\pi}{l}\right)^2\alpha_1 - \alpha_2\right)\Omega(n) = 0 \quad (13)$$

Let $\dfrac{n\pi}{l} = \lambda$ (14)

Then Equation (13) becomes:

$$(\lambda^4 + \lambda^2\alpha_1 - \alpha_2)\Omega(n) = 0 \quad (15)$$

For non-trivial solutions, $\Omega(n) \neq 0$, the characteristic buckling equation is the fourth degree polynomial in λ given by:

$$\lambda^4 + \lambda^2\alpha_1 - \alpha_2 = 0 \quad (16)$$

Solving, we obtain the roots as:

$$\lambda^2 = \frac{-\alpha_1 \pm \sqrt{(\alpha_1^2 - 4(-\alpha_2))}}{2} = \frac{-\alpha_1 \pm \sqrt{(\alpha_1^2 + 4\alpha_2)}}{2} \quad (17)$$

Thus,

$$\lambda^2 = \frac{-\alpha_1 - \sqrt{(\alpha_1^2 + 4\alpha_2)}}{2} \quad (18)$$

$$\lambda^2 = \frac{-\alpha_1 + \sqrt{(\alpha_1^2 + 4\alpha_2)}}{2} \quad (19)$$

Hence, the four roots of the characteristic buckling equation are found as:

$$\lambda = \pm\sqrt{\left(\frac{-\alpha_1 - \sqrt{(\alpha_1^2 + 4\alpha_2)}}{2}\right)} = \pm i\beta_1 \quad (20)$$

$$\lambda = \pm\sqrt{\left(\frac{-\alpha_1 + \sqrt{(\alpha_1^2 + 4\alpha_2)}}{2}\right)} = \pm\sqrt{\left(\frac{\sqrt{(\alpha_1^2 + 4\alpha_2)} - \alpha_1}{2}\right)} = \pm\beta_2 \quad (21)$$

where,

$$\beta_1 = \sqrt{\left(\frac{\alpha_1 + \sqrt{(\alpha_1^2 + 4\alpha_2)}}{2}\right)} \quad (22)$$

$$\beta_2 = \sqrt{\left(\frac{\sqrt{(\alpha_1^2 + 4\alpha_2)} - \alpha_1}{2}\right)} \quad (23)$$

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RESULTS

The buckling moments are obtained using the eigenvalues as follows:

$$\lambda = \frac{n\pi}{l} = \pm\beta_2 = \pm\sqrt{\left(\frac{\sqrt{(\alpha_1^2 + 4\alpha_2)} - \alpha_1}{2}\right)} \quad (24)$$

Squaring both sides, we have:

$$\left(\frac{n\pi}{l}\right)^2 = \frac{\sqrt{(\alpha_1^2 + 4\alpha_2)} - \alpha_1}{2} \quad (25)$$

Simplifying,

$$\alpha_1^2 + 4\alpha_2 = \left(\alpha_1 + 2\left(\frac{n\pi}{l}\right)^2\right)^2$$
$$= \alpha_1^2 + 4\alpha_1\left(\frac{n\pi}{l}\right)^2 + 4\left(\frac{n\pi}{l}\right)^4 \quad (26)$$

Further simplification yields:

$$4\alpha_2 = 4\alpha_1\left(\frac{n\pi}{l}\right)^2 + 4\left(\frac{n\pi}{l}\right)^4 \quad (27)$$

Hence,

$$\alpha_2 = \left(\frac{n\pi}{l}\right)^2\left(\alpha_1 + \left(\frac{n\pi}{l}\right)^2\right) = \frac{M_0^2}{EI_w EI_y} \quad (28)$$

Thus,

$$M_0^2 = EI_w EI_y\left(\frac{n\pi}{l}\right)^2\left(\frac{GJ}{EI_w} + \left(\frac{n\pi}{l}\right)^2\right) \quad (29)$$

or,

$$M_0^2 = \left(\frac{n\pi}{l}\right)^2 EI_y\left(GJ + \left(\frac{n\pi}{l}\right)^2 EI_w\right) \quad (30)$$

The buckling moments are obtained for the buckling modes as:

$$M_0 = \sqrt{\left\{\left(\frac{n\pi}{l}\right)^2 EI_y\left(GJ + \left(\frac{n\pi}{l}\right)^2 EI_w\right)\right\}} \quad (31)$$

Also from Equation (20) we have:

$$\lambda = \frac{n\pi}{l} = \pm\sqrt{\left(\frac{-\alpha_1 - \sqrt{(\alpha_1^2 + 4\alpha_2)}}{2}\right)} \quad (32)$$

Squaring,

$$\left(\frac{n\pi}{l}\right)^2 = \frac{-\alpha_1 - \sqrt{(\alpha_1^2 + 4\alpha_2)}}{2} \quad (33)$$

Simplifying,

$$-\alpha_1 - \sqrt{(\alpha_1^2 + 4\alpha_2)} = 2\left(\frac{n\pi}{l}\right)^2 \quad (34)$$

Further simplification yields

$$-\sqrt{(\alpha_1^2 + 4\alpha_2)} = 2\left(\frac{n\pi}{l}\right)^2 + \alpha_1 \quad (35)$$

This Equation (35) yields the same result for the elastic buckling load (moment) which is given as Equation (31).

Critical buckling moment M_{0cr}

The lowest value of the buckling moment for bi-symmetrical cross-section beams occurs from the expression for M_0 when n = 1. The critical buckling moment M_{0cr} for the case of beams with bi-symmetric cross-sections, under uniform moment is thus obtained as:

$$M_{0cr} = \sqrt{\left\{\frac{\pi^2 EI_y}{l^2}\left(\frac{\pi^2 EI_w}{l^2} + GJ\right)\right\}} \quad (36)$$

$$M_{0cr} = \sqrt{(P_{Eyy} P_\phi \overline{r}_0^2)} \quad (37)$$

$$\text{where } P_{Eyy} = \frac{\pi^2 EI_y}{l^2} \quad (38)$$

$$P_\phi \overline{r}_0^2 = \frac{\pi^2 EI_w}{l^2} + GJ \quad (39)$$

P_{Eyy} is the Euler (flexural) buckling load about the minor axis
P_ϕ is the torsional buckling load
$\overline{r}_0$ is the polar radius of gyration of the cross-section about the shear centre.

DISCUSSIONS

The finite Fourier sine transform method has been successfully used in this work as an analytical tool to solve the boundary value problem (BVP) of the elastic stability under uniform bending moment of a thin-walled beam with both ends $(z = 0, z = l)$ pinned. The governing equation is represented by the fourth order ordinary differential equation (ODE) given as Equation (1) for beams with longitudinal axis defined by the z axis. The finite Fourier sine transformation method is ideally suited to the solution of the BVP since the integral kernel function satisfies the Dirichlet boundary conditions at the pinned ends. Application of the finite Fourier sine transformation to the governing domain equation gave the integral equation given as Equation (8). The linearity properties of the finite Fourier sine transform was used to express the integral equation in algebraic form as Equation (13). Equation (13) which is a homogeneous equation is an algebraic eigenvalue problem. The characteristic buckling equation was obtained for non-trivial solutions as the fourth degree polynomial given by Equation (16). The four eigenvalues (zeros) of the fourth degree polynomial were obtained as Equations (20) and (21).

The buckling moments for the *n* buckling modes were found from the eigenvalues as Equation (31). The critical buckling moment was found to correspond to the first buckling mode and was obtained as Equation (36), which was expressed in terms of the flexural and torsional buckling loads as Equation (37). The expressions obtained for the elastic buckling moment and the critical elastic buckling moment agreed with the solutions presented in literature by researchers who used various other methods.

CONCLUSIONS

The following conclusions are made:

a) The finite Fourier sine transformation method is an ideal analytical tool for the analysis of the elastic buckling moments of thin-walled beams (under uniform moments) with pinned ends at $z = 0$, $z = l$ because the integral kernel function satisfies the Dirichlet boundary conditions at the pinned ends.

b) The finite Fourier sine transform method simplified the boundary value problem by transforming the governing domain ODE to an algebraic eigen value problem represented/given by an algebraic homogeneous equation.

c) The conditions for non-trivial solutions gave the characteristic buckling equation as a fourth degree polynomial equation.

d) The eigenvalues were found as the roots (zeros) of the characteristic polynomial (equation), and were used to find the *n* buckling moments for the *n* buckling modes.

e) The critical buckling moment was obtained as the minimum of the buckling moments and occurred at the first buckling mode.

NOMENCLATURE

M_0 uniform (applied) bending moment
E Young's modulus of elasticity
I_w Saint Venant warping constant or warping constant
EI_y flexural rigidity
G shear modulus or the modulus of rigidity
J Saint Venant torsion constant
I_y moment of inertia
$\phi(z)$ twist (rotational) displacement
z longitudinal coordinate axis of the beam
l unbraced length of the beam
α_1 parameter defined in terms of GJ, and EIw
α_2 parameter defined in terms of M_0, EI_w and EI_y
n integer
Ω (n) finite Fourier sine transform of $\phi(z)$
λ parameter defined in terms of n, l and π
β_1, β_2 parameters defining the roots of the characteristic buckling equation
P_{Eyy} Euler (flexural) buckling load about the minor axis
P_ϕ torsional buckling load
$\overline{r}_0$ polar radius of gyration about the shear centre
$\int$ integration

Subscripts

cr critical
xy Cartesian coordinate plane of the cross-section

Superscript:

$''$ second derivative with respect to z
iv fourth derivative with respect to z

$$\phi''(z) = \frac{d^2\phi(z)}{dz^2}$$

$$\phi^{iv}(z) = \frac{d^4\phi(z)}{dz^4}$$

Abbreviations

ODE Ordinary differential equation
LTB Lateral torsional buckling
FEM Finite element method
BVP Boundary value problem
FLB Flange local buckling
WLB Web local buckling

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