MODELING OF A BEAM FOUNDATION USING AN IMPROVED WINKLER THEORY AND MAXIMA(GNU)

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ABSTRACT

A local analysis of a foundation, based on Winkler's theory, was performed, in which a foundation line is idealized as a beam supported on uniformly distributed elastic supports. The deformability of the soil is represented by deformable supports idealized as displacement springs, and its stiffness constants depend on both the ground and the foundation used. A method was implemented in which the foundation is idealized as a beam, but unlike the Winkler Theory, the separation (loss of contact) between the ground and the foundation was allowed when normal tensile forces on the foundation were developed, and the interaction of the soil laterally(interaction between the springs) was added. These analyzes were proposed by means of matrix stiffness methods using the formulation of the beam-column element and considerations of the Winkler and Akin theory, implementing a case study in Maxima(GNU) which is contrasted with models performed in the finite element software, SAP2000® and Ansys®, to validate the results. Finally, the results were compared, and the percentage change was established to conclude that it is possible to perform a realistic model of the structural problem of a foundation on elastic basis using matrix stiffness methods based on the beam-column element and the interaction of contact between the two elastic elements.

Keywords: soil foundation, Winkler, interaction structure-soil, maxima(GNU), numerical integration, SAP2000®, Ansys®.

1. INTRODUCTION

In most structures, the foundation is supported on deformable soils, which, in practice, are considered elastic. Typical examples are isolated footing and grade beams.

Therefore, when analyzing a foundation line, it can be idealized as a beam supported on uniformly distributed elastic supports, as shown in Figure-1.

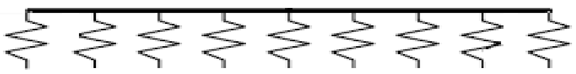

Figure-1. Idealization of a beam with elastic deformable supports.

The stresses of a beam on elastic supports are related to its own deformation while the distribution of pressures in the foundation depends on the relative stiffness of the foundation and its elasticity.

This problem has been extensively studied [1], [2] and most of the solutions that exist are based on Winkler's theory, which proposes that there is permanent contact between the ground and the beam, which means that the soil adheres completely to the beam. However, there are situations in which this assumption is not fulfilled [3] - [9], since the contact between the ground and the foundation can be lost in certain situations due to some static or dynamic loading conditions, such as a normal load of great magnitude applied eccentrically on the foundation or seismic loading [10] - [12].

This paper presents the results of a research, in which the formulation proposed by Akin [3] was adopted, wherein the foundation is idealized as a beam supported on elastic springs allowing separation (loss of contact) between the ground and the foundation, when normal tensile forces develop on the foundation, and the interaction of the soil is introduced laterally (interaction between the springs).

This methodology for modeling foundations on elastic supports achieves reliable results when compared to finite element formulations. The proposed mathematical model was validated with the SAP2000® and Ansys® software, to establish the effectiveness of the implemented method.

2. METHODS

2.1 Differential equation of equilibrium of a beam on elastic supports [3], [7]

In equation (1) is shown the differential equation of equilibrium of a beam on elastic supports, considering the loss of contact between the ground and the foundation when normal tension forces develop on the foundation, as well as the dimension of the areas where contact is lost:

$$EI\frac{d^4w}{dx^4} + N\frac{d^2w}{dx^2} - k_s\frac{d^2w}{dx^2} + k_w w = q(x) \qquad (1)$$

Where the parameter EI is the stiffness bending of the beam, N is the axial tensile force, kw is the Winkler module, *ks* is the second foundation parameter and *q* is the normal load applied to the foundation. Thus, when a normal linear load *q(x)* is applied to the upper surface of the foundation, beam is bending, thereby the foundation resists this action with a linear reaction *p(x).*

The classic model proposed by Winkler assumes that the foundation responds exclusively with a normal reaction *p(x)* to the beam, thereby this reaction is directly proportional to the deformation of the beam, as shown in equation (2).

$$p(x) = k_w w(x) \qquad (2)$$

VOL. 14, NO. 23, DECEMBER 2019 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

In practice, this model does not adequately represent the characteristics of some foundations as it assumes a line of uniformly distributed elastic springs that do not interact with each other. Therefore, several authors have introduced improvements to the classic Winkler model through an additional parameter, k_s, that represents the interaction between the springs (Akin) as shown in equation (3).

$$p(x) = k_w w(x) - k_s \frac{d^2 w(x)}{dx^2} \tag{3}$$

The definition of the stiffness matrix of the element in local coordinates $[k']$ofthe beam shown in Figure-2 is given by equation (4).

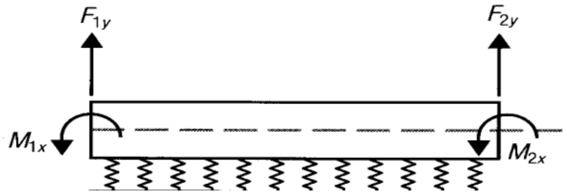

Figure-2. Idealization of the two-dimensional beam element on elastic supports.

$$[k'] = [k_1] + [k_2] + [k_3] + [k_4] \tag{4}$$

Where $[k_1]$ is the conventional stiffness matrix of a beam element, $[k_2]$ is the geometric stiffness matrix that includes the axial load N, $[k_3]$ is the stiffness matrix related to the second foundation parameter k_s and $[k_4]$ is the foundation stiffness matrix of Winkler. For the beam shown in Figure-2, the matrices are as follows:

$$[k_1] = \begin{bmatrix} r_{aax} & r_{abx} & -r_{aax} & r_{bax} \\ r_{abx} & r_{11x} & -r_{abx} & r_{l2x} \\ -r_{aax} & -r_{abx} & r_{aax} & -r_{bax} \\ r_{bax} & r_{21x} & -r_{bax} & r_{22x} \end{bmatrix} \tag{5}$$

$$[k_2] = \begin{bmatrix} r_{g1x} & r_{g2x} & -r_{g1x} & r_{g2x} \\ r_{g2x} & r_{g3x} & -r_{g2x} & -r_{g4x} \\ -r_{g1x} & -r_{g2x} & r_{g1x} & -r_{g2x} \\ r_{g2x} & -r_{g4x} & -r_{g2x} & r_{g3x} \end{bmatrix} \tag{6}$$

$$[k_3] = \frac{k_s}{N}[k_2] \tag{7}$$

$$[k_4] = \begin{Bmatrix} r_{wlx} & r_{w2x} & r_{w4x} & -r_{w5x} \\ r_{w2x} & r_{w3x} & r_{w5x} & -r_{w6x} \\ r_{w4x} & r_{w5x} & r_{wlx} & -r_{w2x} \\ -r_{w5x} & -r_{w6x} & -r_{w2x} & r_{w3x} \end{Bmatrix} \tag{8}$$

Where the coefficients $r_{aax}, r_{abx}, r_{bax}, r_{11x}$ and r_{12x} are the terms of the elastic stiffness matrix that is defined using the flexibilities method.

2.2 Stiffness matrix in local coordinates of a generalized beam-column element [7]

The stiffness matrix in local coordinates of a generalized beam element is defined using the flexibility method, developing a calculation program in Maxima(GNU) Computational Algebra System, which allows solving the problem of numerical incorporation of the terms used in the flexibility matrix, as shown in the equation (9).

$$f] = \begin{Bmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & f_{23} \\ 0 & f_{32} & f_{33} \end{Bmatrix} \tag{9}$$

Where the terms of the flexibility matrix are described by equations (10) to (13).

$$f_{11} = \int_0^l \frac{dz}{EA(z)} \tag{10}$$

$$f_{22} = \int_0^l \frac{z^2 dz}{EI_x(z)} + \int_0^l \frac{dz}{GA_{cy}(z)} \tag{11}$$

$$f_{23} = \int_0^l \frac{z dz}{EI_X(z)} = f_{32} \tag{12}$$

$$f_{33} = \int_0^l \frac{dz}{EI_X(z)} \tag{13}$$

The stiffness matrix is obtained by inverting the flexibility sub-matrices, so its terms are implicitly defined. The global stiffness matrix in local coordinates of the two-column beam-column element of Figure-3 is expressed as showing equation (14).

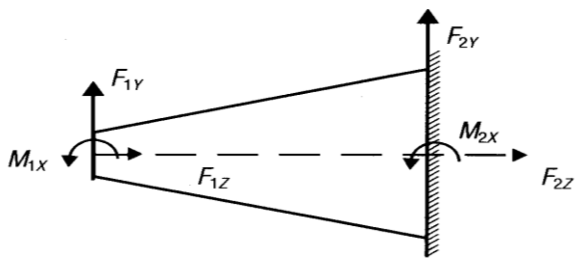

Figure-3. Two-dimensional beam element with a variable section.

$$[K] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{14}$$

The stiffness submatrices are determined by equations (15) to (18).

$$[k_{11}] = \begin{Bmatrix} r_{az} & 0 & 0 \\ 0 & r_{aax} & r_{abx} \\ 0 & r_{abx} & r_{11x} \end{Bmatrix} \tag{15}$$

$$[k_{12}] = \begin{Bmatrix} -r_{az} & 0 & 0 \\ 0 & -r_{aax} & r_{bax} \\ 0 & -r_{abx} & r_{12x} \end{Bmatrix} \tag{16}$$

$$[k_{21}] = [k_{12}]^T \tag{17}$$

$$[k_{22}] = \begin{Bmatrix} r_{az} & 0 & 0 \\ 0 & r_{aax} & -r_{bax} \\ 0 & -r_{bax} & r_{22x} \end{Bmatrix} \quad (18)$$

The seven terms of these stiffness sub matrices are defined by equations (19) to (26).

$$r_{az} = \frac{1}{f_{11}} \quad (19)$$

$$Det_x = f_{22}f_{33} - f_{23}{}^2 \quad (20)$$

$$r_{11x} = \frac{f_{22}}{Det_X} \quad (21)$$

$$r_{12x} = \frac{f_{23}L - f_{22}}{Det_X} \quad (22)$$

$$r_{22x} = \frac{f_{33}L^2 - 2f_{23}L + f_{22}}{Det_X} \quad (23)$$

$$r_{aax} = \frac{r_{11x} + r_{22x} + 2r_{12x}}{L^2} \quad (24)$$

$$r_{abx} = \frac{r_{11x} + r_{12x}}{L} \quad (25)$$

$$r_{bax} = \frac{r_{22x} + r_{12x}}{L} \quad (26)$$

2.3 Foundation stiffness matrices: Matrix related to the second parameter of the foundation $[\boldsymbol{k_3}]$and Winkler matrix $[\boldsymbol{k_4}]$[3], [7]

The coefficients of the geometric stiffness matrix are shown in equations (27) to (30):

$$r_{g1x} = -\frac{36N}{30L} = -\frac{6N}{5L} \quad (27)$$

$$r_{g2x} = -\frac{3NL}{30L} = -\frac{N}{10} \quad (28)$$

$$r_{g3x} = -\frac{4NL^2}{30L} = -\frac{2NL}{15} \quad (29)$$

$$r_{g4x} = -\frac{NL^2}{30L} = -\frac{NL}{30} \quad (30)$$

For the case of the stiffness matrix of Winkler, the coefficients are shown in equations (31) to (36):

$$r_{w1x} = \frac{156k_w}{420} = \frac{39k_w}{105} \quad (31)$$

$$r_{w2x} = \frac{22k_wL}{420} = \frac{11k_wL}{210} \quad (32)$$

$$r_{w3x} = \frac{4k_wL^2}{420} = \frac{k_wL^2}{105} \quad (33)$$

$$r_{w4x} = \frac{54k_w}{420} = \frac{27k_w}{210} \quad (34)$$

$$r_{w5x} = \frac{13k_wL}{420} \quad (35)$$

$$r_{w6x} = \frac{3k_wL^2}{420} = \frac{k_wL^2}{140} \quad (36)$$

In the previous equations, k_w has the units of stiffness (force per unit length) and k_s has units of force. In some references, $k_w = sL_e$ is usually expressed, where s is a contact stress and L_e is an effective length of ground contact with the foundation.

2.4 Matrix system of stiffness equations in local coordinates [7]

Performing a correctly assembling the global stiffness matrix of the foundation structure $[K]$, the system is shown in equation (37).

$$\{F\} = [K]\{u\} \quad (37)$$

Where $\{u\}$ is the vector of the global displacements of the foundation and $\{F\}$ is the vector of external forces applied to the foundation. The matrix system of equations to solve in local coordinates is represented by equation (38).

$$\begin{Bmatrix} \{F_1\} \\ \{F_2\} \end{Bmatrix} = \begin{Bmatrix} [k_{11}][k_{l2}] \\ [k_{21}][k_{22}] \end{Bmatrix} \begin{Bmatrix} \{u_1\} \\ \{u_2\} \end{Bmatrix} \quad (38)$$

Where the force vectors {F} and displacements {u} of the matrix system (38) are defined by equations (39) to (42).

$$\{u_1\} = \begin{Bmatrix} u_{1z} \\ u_{1y} \\ \theta_{lx} \end{Bmatrix} \quad (39)$$

$$\{F_1\} = \begin{Bmatrix} F_{1z} \\ F_{1y} \\ M_{1x} \end{Bmatrix} \quad (40)$$

$$\{u_2\} = \begin{Bmatrix} u_{2z} \\ u_{2y} \\ \theta_{2x} \end{Bmatrix} \quad (41)$$

$$\{F_2\} = \begin{Bmatrix} F_{2z} \\ F_{2y} \\ M_{2x} \end{Bmatrix} \quad (42)$$

Once the element's stiffness matrix is defined in the local coordinates of equation (4), it is incorporated in a calculation system (matrix or finite element software). The stiffness matrix of the element in global coordinates is obtained using transformation matrices, and the connectivity between elements is defined using the assembly rule.

2.5 Mechanical elements [3], [7]

The mechanical elements in the foundation beam and the reaction forces that are transmitted to the continuous elastic area are defined by equation (43).

$$\{F'\} = [k']\{u'\} + \{F_f'\} \quad (43)$$

Where $\{F'\}$ is the vector with the mechanical elements of the beam in local coordinates, $\{u'\}$ is the vector of displacements of the beam in local coordinates, and $\{F_f'\}$ is the vector of the fixing forces of the beam (moments and fixed end reactions) and $[k']$ is the stiffness matrix of the element that includes the stiffness of the beam and the stiffness of the elastic supports, as shown in equation (44).

$$\{F'\} = \left[[k_1] + [k_2] + [k_3] + [k_4]\right]\{u'\} + \{F_f'\} \tag{44}$$

The reactions that are transmitted to the elastic area are calculated using the equation (45):

$$\{F_{cim}'\} = -[[k_3] + [k_4]]\{u'\} \tag{45}$$

Therefore, equation (43) can be rewritten as equation (46).

$$\{F'\} = \left[[k_1] + [k_2]\right]\{u'\} - \{F_{cim}'\} + \{F_f'\} \tag{46}$$

Thus, the reactions transmitted to the foundation and the mechanical elements in the foundation are calculated from equations (43) to (46).

In most footings the local axes of the foundation coincide with the global axes of the foundation system; thereby, $[k'] = [k], \{u'\} = \{u\}$ and $\{F'\} = \{F\}$ since the transformation matrix turns out to be the identity matrix.

2.6 Angles of rotation and fixed end moments of the generalized beam-column element [7]

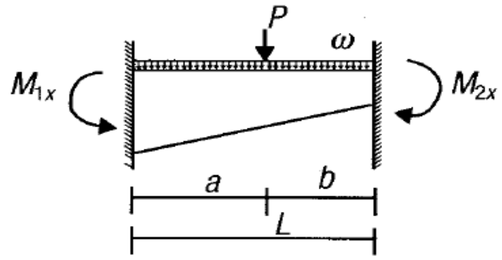

Figure-4. Fixed beam element with loading.

In a fixed beam with a loading condition in its main plane of bending as shown in Figure-4, the angles of rotation can be determined using the conjugate beam method as shown in Figure-5, using an equilibrium equation and considering the shear deformations. The angles of rotation at the ends 1 and 2 of the beam are calculated according to equations (47) and (48).

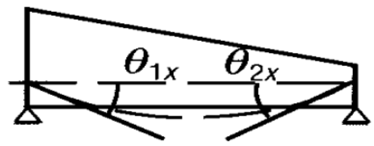

Figure-5. Conjugate beam.

$$\theta_{2x} = \frac{1}{L}\int_0^L \frac{zM_{0x}}{EI_X(z)}dz + \frac{1}{L}\int_0^L \frac{V_{0y}}{GA_{cy}(z)}dz \tag{47}$$

$$\theta_{1x} = \int_0^L \frac{M_{0x}}{EI_X(z)}dz - \theta_{2x} \tag{48}$$

The fixed end moments in the main bending direction are calculated, as shown in Figure-6, using the equations (49) and (50).

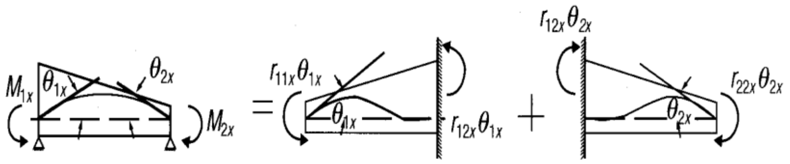

Figure-6. Fixed end moments as a function of the angles of rotation and stiffness of the beam.

$$M_{1x} = r_{11x}\theta_{1x} - r_{12x}\theta_{2x} \tag{49}$$

$$M_{2x} = r_{22x}\theta_{2x} - r_{12x}\theta_{1x} \tag{50}$$

For a uniformly distributed load in the main plane of bending (ω_x), the momentum and shear equations of the isostatic structure are shown in equations (51) and (52).

$$M_{0x} = \frac{\omega_x L}{2}z - \frac{\omega_x z^2}{2} \tag{51}$$

$$V_{0y} = \frac{\omega_x L}{2} - \omega_x z \tag{52}$$

Therefore, the fixing rotations are shown in equations (53) and (54).

$$\theta_{2x} = \frac{\omega_x}{2E}\left[\int_0^L \frac{z^2 dz}{I_x(z)} - \frac{1}{L}\int_0^L \frac{z^3 dz}{I_x(z)}\right] + \frac{\omega_x}{G}\left[\frac{1}{2}\int_0^L \frac{dz}{A_{cy}(z)} - \frac{1}{L}\int_0^L \frac{zdz}{A_{cy}(z)}\right] \tag{53}$$

$$\theta_{1x} = \frac{\omega_x}{2E}\left[L\int_0^L \frac{zdz}{I_x(z)} - \int_0^L \frac{z^2 dz}{I_x(z)}\right] - \theta_{2x} \tag{54}$$

Substituting the fixed angles of rotation (53) and (54) in equations (49) and (50), the fixed end moments of the beam are obtained.

2.7 Mathematical model of grade beam with distributed loading

The calculation algorithm using the software Maxima(GNU) performs the following calculations:

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- Data entry: cross-sectional dimensions, length of sections, *E, G*.
- Statement of variables for each section: $h_w(z)$, $A(z)$, $A_{cy}(z)$, $Y(z)$, $I(z)$.
- Calculate the stiffness coefficients of the matrix $[k_1]$, which are related to the deformation of the beam in local coordinates.
  - Calculation of the integrals of the terms of flexibility and the integrals of the torsion factors for each section: Romberg approximation.
  - Sum of the terms of flexibility of the beam.
  - Sum of the terms of torsion of the beam.
  - Calculation of the fixed angles of rotation for a uniform distributed loading.
  - Determination of the terms of the stiffness matrix based on the flexibility factors.
- Calculate the coefficients for the geometric stiffness matrix $[k_2]$ in local coordinates.
- Calculate the coefficients of the matrix $[k_3]$related to the second parameter of the foundation, *ks*.
- Calculate the stiffness matrix of each element in local coordinates: $[k^{'}] = [k_1] + [k_2] + [k_3] + [k_4]$.
- Calculate the foundation stiffness matrix as: $\{k'_{cim}\} = -[[k_3] + [k_4]]$.
- Separate the stiffness sub-matrices of each element into local coordinates.
- Assemble the stiffness matrix of the beam. (using the assembly rule to obtain the stiffness matrix in global coordinates from the stiffness sub-matrices of each element).
- Calculate the fixed end moments for a uniform distributed loading ω_x.
- Calculate the shear strength at the ends of the beam V_{1y} and V_{2y}.
- Assembly of the fixed end forces vector.
- Stiffness matrix equation: [Fn-FFn]=[Knn][δn].
- Determination of the displacements [δn]
- Matrix stiffness equation: Reaction forces: [Fa]=[Kan][δn]+ [FFa].
- Calculation of the foundation and mechanical elements reaction forces.
  - Calculate the foundation reaction forces for each element: $\{F^{'}_{cim}\} = -[k^{'}_{cim}]\{u^{'}\}$.
  - Calculate the mechanical elements for each section: $\{F^{'}\} = [k^{'}]\{u^{'}\} + \{F^{'}_f\}$..

In Figure-7, a grade beam of a rectangular section, with a uniformly distributed loading in its length and supported on an elastic foundation, is showed. The beam model is represented by a discretization of four elements of equal length (shear deformations were not considered).

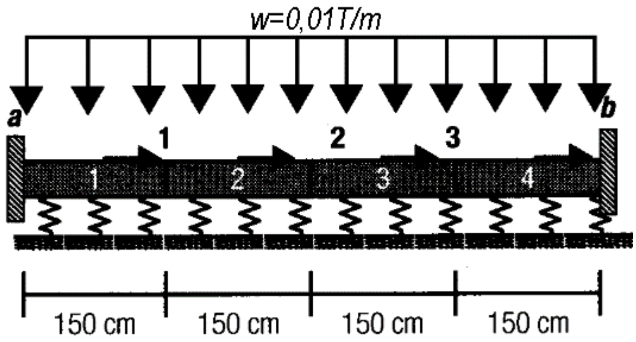

Figure-7. Grade beam with a distributed loading.

The cross section of the beam is *30 × 90 cm*, the modulus of elasticity of the beam is E = 200 Ton/cm^2, the Winkler stiffness module per unit length is $k_w = 5$ Ton/cm, $k_s = 20$ Ton, $N = 5$ Ton. The normal load applied to the beam is *w*=0, 01 Ton/*m*, and the geometric properties of the beam are $A = 2{,}700 cm^2$ and $I = 1{,}822{,}500 cm^4$.

According to the discretization of the beam illustrated in Figure-7, and since the beam is restricted from movement at its ends (fixed), the degrees of freedom of the vector $\{u\}$ are:

$$\{u\} = \begin{Bmatrix} u_{Jy} \\ \theta_{1x} \\ u_{2y} \\ \theta_{2x} \\ u_{3y} \\ \theta_{3x} \end{Bmatrix};$$

Since the sections are equal and the local axes coincide with the global ones, only the stiffness matrices in local coordinates should be obtained. For beams on elastic supports, the stiffness matrix in local coordinates is calculated as: $[k'] = [k_1] + [k_2] + [k_3] + [k_4]$. Therefore:

$[k_1^1] = [k_1^2] = [k_1^3] = [k_1^4]$.

$[k_2^1] = [k_2^2] = [k_2^3] = [k_2^4]$.

$[k_3^1] = [k_3^2] = [k_3^3] = [k_3^4]$.

$[k_4^1] = [k_4^2] = [k_4^3] = [k_4^4]$.

Generally: $[k'^1] = [k'^2] = [k'^3] = [k'^4]$.

On the other hand, the foundation matrix is, $\{k'_{foun}\} = -[[k_3] + [k_4]]$

Hence: $[k'_{foun}{}^1] = [k'_{foun}{}^2] = [k'_{foun}{}^3] = [k'_{foun}{}^4]$.

And the stiffness sub-matrices of each element in local coordinates, used to apply the assembly rule, are:

$[k_{11}^1] = [k_{11}^2] = [k_{11}^3] = [k_{11}^4]$.

$[k_{12}^1] = [k_{12}^2] = [k_{12}^3] = [k_{12}^4]$.

$[k_{21}^1] = [k_{21}^2] = [k_{21}^3] = [k_{21}^4]$.

$[k_{22}^1] = [k_{22}^2] = [k_{22}^3] = [k_{22}^4]$.

The fixed forces vector $\{F_f\}$ of each element is:

$$\{F_f\} = \begin{Bmatrix} F_{f1y} \\ M_{f1x} \\ F_{f2y} \\ M_{f2x} \\ F_{f3y} \\ M_{f3x} \end{Bmatrix} = \begin{Bmatrix} 150 \\ 0 \\ 150 \\ 0 \\ 150 \\ 0 \end{Bmatrix} \text{(Ton)}$$

and, $\{F\} - \{F_f\} = [K]\{u\}$. The global displacement vector is,

$$\{u\} = \begin{Bmatrix} u_{1y} \\ \theta_{\theta_{1x}} \\ u_{2y} \\ \theta_{2x} \\ u_{3y} \\ \theta_{3x} \end{Bmatrix}.$$

The reaction forces of the foundation and the mechanical elements are calculated from: $\{F'_{foun}\} = -[k'_{foun}]\{u'\}$. Therefore,$\{F'\} = [[k_1] + [k_2]]\{u'\} - \{F'_{foun}\} + \{F'_f\}$, or simply, $\{F'\} = [k']\{u'\} + \{F'_f\}and\{F\} - \{F_f\} = [K]\{u\}$.

Element 1

$$\{u^1\} = \{u'^1\} = \begin{Bmatrix} \{u_A\} \\ \{u_1\} \end{Bmatrix} = \begin{Bmatrix} u_{Ay} \\ \theta_{Ax} \\ u_{ly} \\ \theta_{lx} \end{Bmatrix}.$$

$$\{F'^1_{foun}\} = -[k'^1_{found}]\{u'^1\} = \begin{Bmatrix} F'^1_{Ayfoun} \\ M'^1_{Axfoun} \\ F'^1_{1yfoun} \\ M'^1_{1xfound} \end{Bmatrix}.$$

$$\{F'^1\} = [k'^1]\{u'^1\} + \{F'^1_f\} = \begin{Bmatrix} F'^1_{Ay} \\ M'^1_{Ax} \\ F'^1_{1y} \\ M'^1_{1x} \end{Bmatrix}.$$

Element 2

$$\{u^2\} = \{u'^2\} = \begin{Bmatrix} \{u_1\} \\ \{u_2\} \end{Bmatrix} = \begin{Bmatrix} u_{1y} \\ \theta_{1x} \\ u_{2y} \\ \theta_{2x} \end{Bmatrix}.$$

$$\{F'^2_{foun}\} = -[k'^2_{foun}]\{u'^2\} = \begin{Bmatrix} F'^2_{1yfoun} \\ M'^2_{1xfoun} \\ F'^2_{2yfoun} \\ M'^2_{2xfoun} \end{Bmatrix}.$$

$$\{F'^2\} = [k'^2]\{u'^2\} = \begin{Bmatrix} F'^2_{1y} \\ M'^2_{1x} \\ F'^2_{2y} \\ M'^2_{2x} \end{Bmatrix}.$$

Element3

$$\{u^3\} = \{u'^3\} = \begin{Bmatrix} \{u_2\} \\ \{u_3\} \end{Bmatrix} = \begin{Bmatrix} u_{2y} \\ \theta_{2x} \\ u_{3_y} \\ \theta_{3x} \end{Bmatrix}.$$

$$\{F'^3_{cim}\} = -[k'^3_{cim}]\{u'^3\} = \begin{Bmatrix} F'^3_{2yfoun} \\ M'^3_{2xfoun} \\ F'^3_{3yfoun} \\ M'^3_{3xfoun} \end{Bmatrix}.$$

$$\{F'^3\} = [k'^3]\{u'^3\} = \begin{Bmatrix} F'^3_{2y} \\ M'^3_{2x} \\ F'^3_{3y} \\ M'^3_{3x} \end{Bmatrix}.$$

Element4

$$\{u^4\} = \{u'^4\} = \begin{Bmatrix} \{u_3\} \\ \{u_B\} \end{Bmatrix} = \begin{Bmatrix} u_{3y} \\ \theta_{3x} \\ u_{By} \\ \theta_{Bx} \end{Bmatrix}.$$

$$\{F'^4_{cim}\} = -[k'^4]\{u'^4\} = \begin{Bmatrix} F'^4_{3yfoun} \\ M'^4_{3xfoun} \\ F'^4_{Byfoun} \\ M'^4_{Bxfoun} \end{Bmatrix}.$$

$$\{F'^4\} = \begin{Bmatrix} F'^4_{3y} \\ M'^4_{3x} \\ F'^4_{By} \\ M'^4_{Bx} \end{Bmatrix}.$$

2.8 SAP2000®and Ansys® Modeling

The modeling of the foundation beam problem in the Ansys®software is developed using the finite elements BEAM3 and COMBIN14.

BEAM3 is based on Timoshenko's theory of beams and includes shear deformations effects. It is a two-node linear beam element that has three degrees of freedom in each node: translations in the nodal directions x, y and torsion around the nodal axis z. The element is defined by the two nodes, the cross-sectional area, the moment of inertia of the area, the height and the properties of the material.

VOL. 14, NO. 23, DECEMBER 2019 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

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COMBIN14is a damping element that has longitudinal and torsional capacity in 1-D, 2-D or 3-D applications. When it acts as a longitudinal spring damper, it is a uniaxial compression element with up to three degrees of freedom in each node: translations in the nodal directions *x*, *y* and *z*. The element does not consider bending or rotation. When it acts as a torsional spring damper, it is a rotating element with three degrees of freedom in each node: rotations around the *x*, *y* and *z* nodal axes. The element is defined by two nodes, a spring constant (k) and two damping coefficients (cv). The longitudinal spring constant must have units of Force/Length.

The finite element model of the foundation beam using Ansys®is shown in Figure-8 and Figure-9.

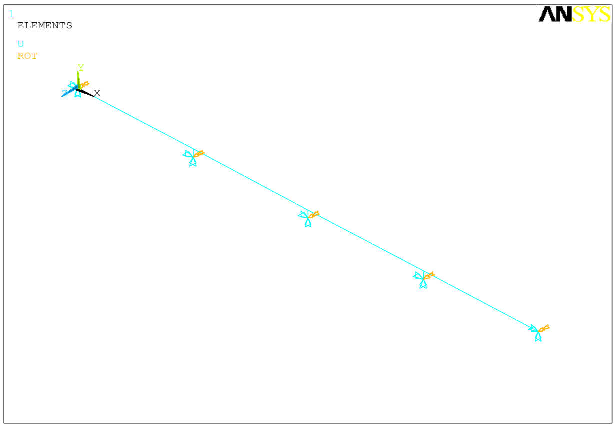

Figure-8. Grade beam model using Ansys®.

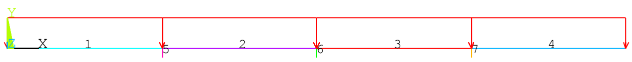

Figure-9. Beam model using Ansys®and loading.

Using SAP2000®, the T-beam model is made with a two-node "frame" bar element that includes the damping effect. The distributed load is applied in the length of the beam, and performing a static analysis, the degrees of freedom and the reaction forced in the nodes of the ends and the internal forces in the element are obtained.

The finite element model of the beam on the elastic foundation developed using SAP2000®is shown in Figure-10.

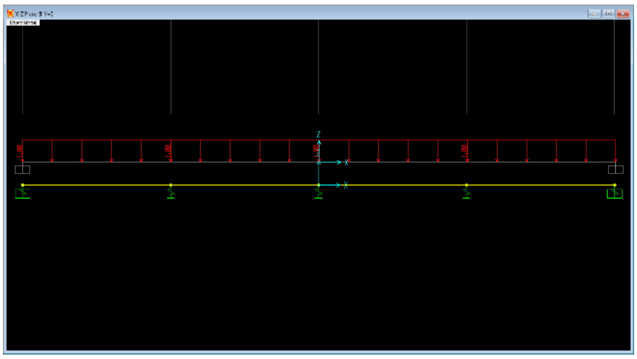

Figure-10. Beam model with springs using SAP2000®.

3. RESULTS AND DISCUSSIONS

The integrals of the matrix formulation were determined and then developed in a calculation software, the free CAS Maxima(GNU), in order to solve the problem of the beam on an elastic foundation subject to uniform distributed loading (ω_x). The results of the mathematical model developed using Maxima(GNU) are compared with identical models of finite elements using the *Ansys®* and *SAP2000®* software, in order to find similarity that leads us to conclude that the mathematical calculation of integration of flexibility matrices and the improvements introduced to the original Winkler model, are an economical and efficient option for the analysis of beams on an elastic foundation.

The calculation code developed allows to consider the shear deformations and the haunch (taper beam). On the other hand, the Romberg integration technique was used to calculate the flexibility and rotation factors.

As shown in Figure-11 and Figure-12, the illustrated results of the foundation reaction forces, and the mechanical elements are obtained from the flexibility model developed using Maxima(GNU). The units of the forces are Newton and the units of length are centimeters.

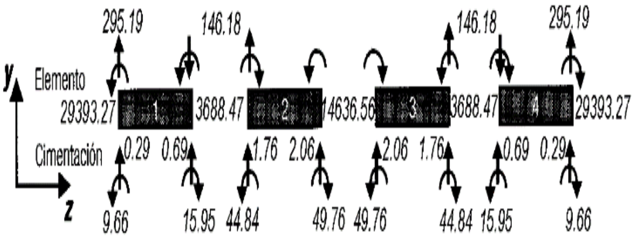

Figure-11. Free body diagram of the beam divided by elements.

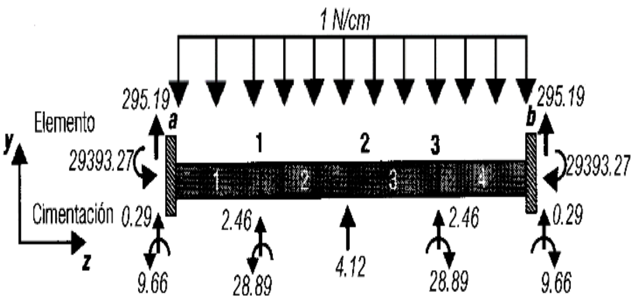

Figure-12. Free body diagram of the beam.

The foundation reaction forces obtained from the flexibility model developed using Maxima(GNU), and the finite element models using Ansys® and SAP2000® software are compared as shown in Table-1 and Table-2, in order to validate the matrix formulation used. The variation in the reaction forces of the foundation is minimal. The improved Winkler formulation allows to know the moment of the reaction forces of the foundation, this situation can only be validated when the torsional stiffness of the springs in the direction of bending is known, in this case, it was estimated as -6387.35 Ton.

ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

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Table-1. Reaction forces in the foundation.

	Force - Reaction[N]			
NODE	**ANSYS**	**SAP2000**	**MAXIMA**	**VAR.(%)**
1	---	---	0.29	---
2	2.545	2.54	2.46	3.34
3	4.521	4.52	4.12	8.87
4	2.545	2.54	2.46	3.34
5	---	---	0.29	---

Table-2. Reaction moments in the foundation.

	Moment - Reaction[N-cm]			
NODE	**ANSYS**	**SAP2000**	**MAXIMA**	**VAR.(%)**
1	---	---	9.66	---
2	---	---	28.89	---
3	---	---	0	---
4	---	---	-28.89	---
5	---	---	-9.66	---

The mechanical elements obtained from the flexibility model developed in Maxima(GNU) and the finite element models using Ansys® and SAP2000® are compared in Tables 3 to 6, in order to validate the matrix formulation implemented. As shown, the variation of the results does not exceed 1%.

Table-3. Results of moment at node i of the beam.

	Moment-i [N-cm]			
ELEM	**ANSYS**	**SAP2000**	**MAXIMA**	**VAR.(%)**
1	-29375	-29374.67	-29393.27	0.06
2	3654.6	3654.57	-3688.47	0.93
3	14566	14565.53	-14636.56	0.48
4	3654.6	3654.57	-3688.47	0.93

Table-4. Results of moment at node j of the beam.

	Moment-j [N-cm]			
ELEM	**ANSYS**	**SAP2000**	**MAXIMA**	**VAR.(%)**
1	3654.6	3654.57	3688.47	0.93
2	14566	14565.53	14636.56	0.48
3	3654.6	3654.57	3688.47	0.93
4	-29375	-29374.67	-29393.27	0.06

Table-5. Results of shearing forces at node i of the beam.

	Shear-i [N]			
ELEM	**ANSYS**	**SAP2000**	**MAXIMA**	**VAR.(%)**
1	-295.19	-295.20	295.19	0.00
2	-147.74	-147.74	146.18	1.05
3	-2.2603	-2.26	0.00	---
4	145.19	145.20	-146.18	0.68

Table-6. Results of shearing forces at node j of the beam.

	Shear-j [N]			
ELEM	**ANSYS**	**SAP2000**	**MAXIMA**	**VAR.(%)**
1	-145.19	-145.20	-146.18	0.68
2	2.2603	2.26	0.00	---
3	147.74	147.74	146.18	1.05
4	295.19	295.20	295.19	0.00

The results of the modeling of the beam on the elastic foundation using Ansys® and SAP2000® are shown from Figure-13 to Figure-19. These results are equal to the results of the improved Winkler model, therefore, the accuracy of the method implemented using Maxima(GNU) is verified.

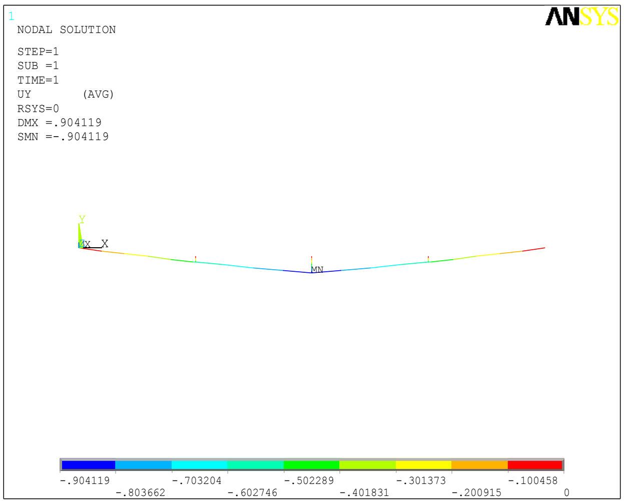

Figure-13. Isocontour of displacement in Y, using Ansys®.

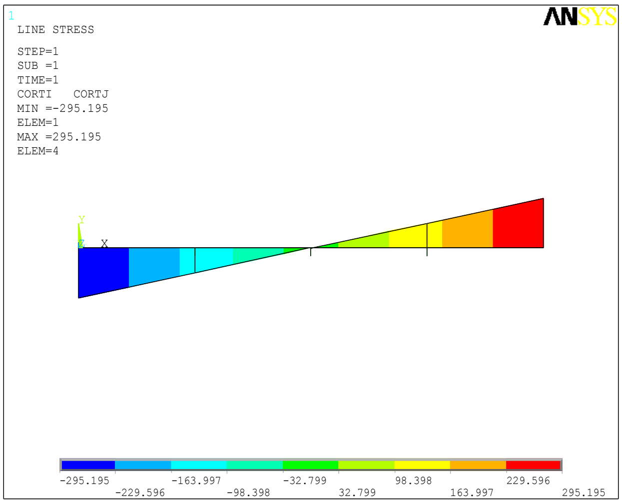

Figure-14. Isocontour of the shear force using Ansys®.

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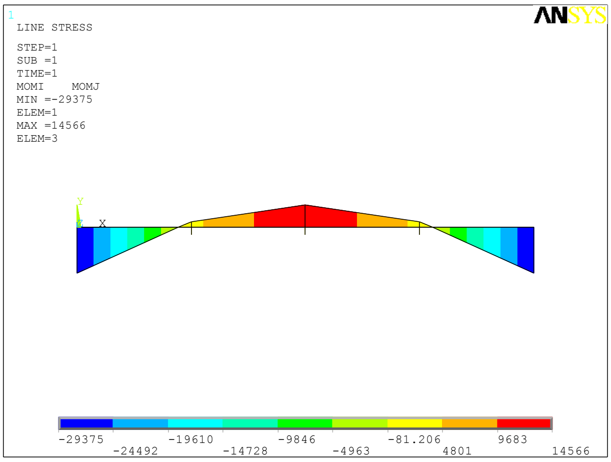

Figure-15. Isocontour of moments - Ansys®.

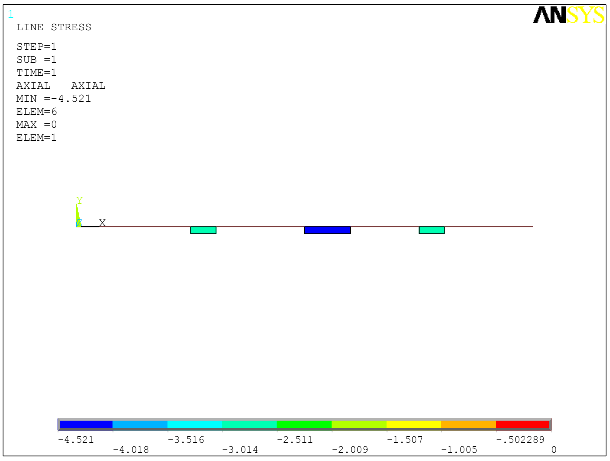

Figure-16. Isocontour Axial forces of the springs - Ansys®.

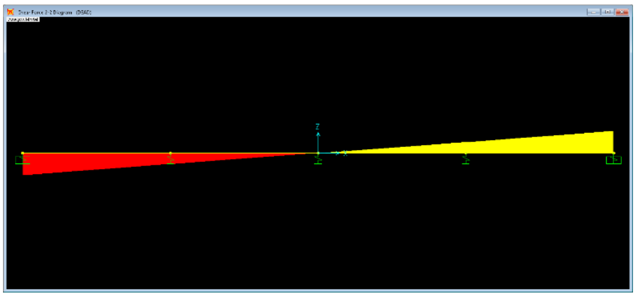

Figure-17. Results of the shear force - SAP2000®.

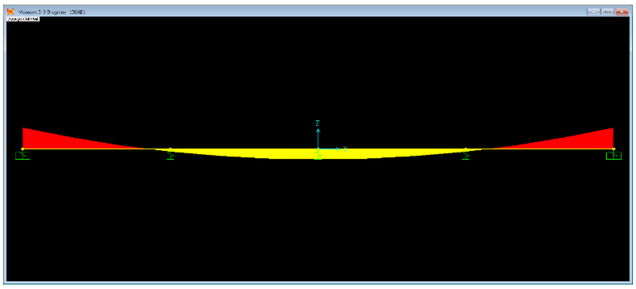

Figure-18. Results of the moment - SAP2000®.

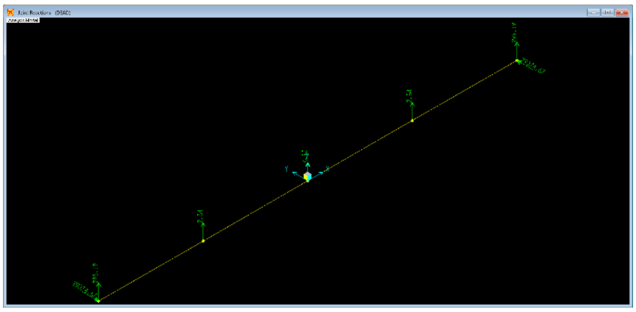

Figure-19. Reaction forces on the foundation - SAP2000®.

4. CONCLUSIONS

- The matrix calculation system developed in Maxima(GNU) based on the numerical integrals of the terms of flexibility, and the improved Winkler theory shows similar results to the results obtained with Ansys® and Sap2000® for beams on elastic foundations, which allowed validating the accuracy of the model.
- The calculation code developed integrates elements of beams of variable cross section, and shear deformations, which makes it very efficient when performing these types of analysis.
- The improved Winkler theory allows to know the moments of the reaction forces of the foundation, a situation that can only be validated when the torsional stiffness of the springs in the direction of bending is known, which in this case, it was estimated as -6387.35 Ton.
- When using this type of model, it is very important to make an accurate discretization, using a reasonable number of elements to represent the foundation, in order to obtain a reliable response, especially if the foundation is considered displaced in a semi-space without lateral boundaries that fix its deformation. .
- It was possible to demonstrate that a methodology of a classical calculation of a systematic and reasonable implementation, based on numerical integration and interaction theories, can equal the results obtained from robust finite element software, becoming a reliable and accurate option.
- The effectiveness and precision of the model developed were verified, showing significant advantages in the analysis of beams on an elastic foundation.
- The calculation possibilities that computer programs offer us nowadays and the advances in mathematical calculation techniques, leads to know that it is possible to solve problems of great importance in structural engineering without the expensive use of commercial software of finite elements.

ACKNOWLEDGMENT

We thank Surcolombiana University for the support given to this research (Project number 3085). This paper is the result of one of the researches developed in the “Model-ing Structural” group led by the first author.

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