APPROXIMATION TO THE COVARIANCE MATRIX FOR STOCHASTIC POINT KINETICS

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ABSTRACT

In this work, the square root of the matrix of variances in stochastic point kinetics is analytically deduced using Cholesky decomposition. The system under study is solved numerically using the implicit Milstein scheme, the variance of the neutron population density and the concentration of precursors can be reduced, and better approximations of the expected values are obtained through the implementation of a new independent Brownian motion. The results obtained comprise different configurations of the reactivity parameters, precursor population, time steps, Brownian motion and initial conditions. The results were compared with those reported in the literature, being consistent with them, which is a manifestation of the efficiency of the proposed method.

Keywords: cholesky decomposition, stochastic point kinetics, nuclear reactors, implicit Milstein scheme, neutron population.

INTRODUCTION

Stochastic point kinetics was first introduced using the SPCA (Stochastic Piecewise Constant Approximation) and MC (Monte Carlo) methods [1], in this publication there is a matrix formulation consisting of the product of the square root of the variance matrix and a vector of independent Brownian motion. Later works used the same covariance matrix but using the EM (Euler-Maruyama) method and the T 1.5 (Taylor 1.5) method [2,3], in a subsequent work -without calculating the covariance matrix- a Markov process is assumed to obtain a form called SSPK (Simplified Stochastic Point Kinetics Equations) [4]. Subsequently, other methods were considered making different approaches in the covariance matrix AEM (Analytical Exponential Model) [5], Double DDM (Double Diagonalization-Decomposition Method) [6], ESM (Efficient Stochastic Model) [7], IEM (Implicit Euler-Maruyama) [8], and the recent article published Milstein method from Itô Lemma [9].

In this work, the analytical calculation of the variance matrix is performed using the Cholesky decomposition [10], subsequently, the elements of the square root of the variance matrix are simplified in order to reduce the variance values of the populations of neutrons and precursors, which generates a better approximation in the calculation of expected values. The results obtained have been calculated using the implicit Milstein scheme, these in turn will be compared with those reported in the literature.

THEORETICAL ASPECTS

Stochastic point kinetic equations

As mentioned earlier, stochastic point kinetics was first introduced in [1], the matrix expression deduced by these authors is:

$$\frac{d}{dt}|P(t)\rangle = \hat{A}|P(t)\rangle + |Q(t)\rangle + \hat{B}^{1/2}\frac{d}{dt}|\omega(t)\rangle \qquad (1)$$

Where $|P(t)\rangle$ is the vector of random variables defined by the equation (2)which accounts for the populations of neutrons and precursors, $\hat{A}$ is the matrix of expected values defined in the equation (3), $|Q(t)\rangle$ is the source vector defined in the equation (4), $\hat{B}$ is the matrix of variances defined in the equation (5), $|\omega(t)\rangle$ is the Wiener process vector defined in the equation (6)or by $\sqrt{\Delta t}|\eta\rangle$, these processes are continuous time stochastic with statistically independent and stationary increases, not differentiable in time

$$|P(t)\rangle = \begin{bmatrix} n(t) \\ C_1(t) \\ C_2(t) \\ \vdots \\ C_m(t) \end{bmatrix} \qquad (2)$$

$$\hat{A} = \begin{bmatrix} \frac{\rho(t)-\beta}{\Lambda} & \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & \cdots & 0 \\ \frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_m}{\Lambda} & 0 & 0 & \cdots & -\lambda_m \end{bmatrix} \qquad (3)$$

$$|Q(t)\rangle = \begin{bmatrix} q(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (4)$$

$$\hat{B} = \begin{bmatrix} \xi & a_1 & a_2 & \dots & a_m \\ a_1 & r_1 & b_{2,3} & \dots & b_{2,m} \\ a_2 & b_{3,2} & r_2 & \dots & b_{3,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_m & b_{m,2} & b_{m,3} & \dots & r_m \end{bmatrix} \qquad (5)$$

$$|\omega(t)\rangle = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_{m+1} \end{bmatrix} \quad (6)$$

Where n is the density of neutrons, C_m is the concentration of precursors of the *m-th* group, ρ is the reactivity which accounts for the production of neutrons, β is the total fraction of precursors of delayed neutrons, Λ is the average time of neutron generation, λ_m is the decay constant of the m-class of delayed neutron precursors, q is the magnitude of the external source of neutrons, $\omega_m$are the Wiener processes that are characterized by being stochastic processes of continuous time and independent stationary increases. The elements of the matrix $\hat{B}$are described below:

$$\xi = \gamma n(t) + \sum_{i=1}^{m} \lambda_i C_i(t) + q(t) \quad (7)$$

$$\xi\gamma = \frac{-1-\rho+\upsilon(1-\beta)^2+2\beta}{\Lambda} \quad (8)$$

$$a_m = \frac{\beta_m[\nu(1-\beta)-1]}{\Lambda} n(t) - \lambda_m C_m(t) \quad (9)$$

$$r_m = \frac{\nu\, \beta_m^2}{\Lambda} n(t) + \lambda_m C_m(t) \quad (10)$$

$$b_{i,j} = \frac{\nu \beta_{i-1}\beta_{j-1}}{\Lambda} n(t) \quad (11)$$

Where υ is the average number of neutrons generated per fission event.

Square root of the varianza matrix

The matrix of variances given by equation (5) is Hermitic [10] and real and since every real Hermitic matrix is diagonalizable, its eigenvalues are real and its eigenvectors are orthogonal. If $\hat{B}$ is a real Hermitic matrix, then its square root $\hat{B}^{1/2}$is also real Hermitic. This makes it possible to apply the Cholesky decomposition. That is, write $\hat{B}$ as the product of the transpose of an upper triangular matrix by the upper triangular matrix, as follows:

$$\hat{B} = \hat{R}^T \hat{R} \quad (12)$$

Using the property $\left(\hat{C}^T\right)^n = \left(\hat{C}^n\right)^T$we get

$$\hat{B}^{1/2} = \hat{R}^{1/2\,T} \hat{R}^{1/2} \quad (13)$$

Where

$$\hat{R} = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} & \cdots & r_{1,j} \\ 0 & r_{2,2} & r_{2,3} & \cdots & r_{2,j} \\ 0 & 0 & r_{3,3} & \cdots & r_{3,j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{i,j} \end{bmatrix} \quad (14)$$

In this way, the calculation of the square root of the variance matrix is reduced to the calculation of the square root of a simpler matrix $(\hat{R})$.

Performing the calculation suggested in equation (12), we find that the elements of the matrix $\hat{R}$are:

$$\begin{aligned} r_{1,1} &= \sqrt{\xi} \\ r_{1,i} &= \frac{a_{i-1}}{\sqrt{\xi}} \qquad ; \qquad i = 2,3,4,\ldots \\ r_{i,i} &= \sqrt{r_{i-1} - \sum_{k=1}^{i-1} r_{k,i}^2} \qquad ; \qquad i = 2,3,4,\ldots \\ r_{i,j} &= \frac{1}{r_{i,i}}\left(b_{i,j} - \sum_{k=1}^{i-1} r_{k,i} r_{k,j}\right) \qquad ; \quad i < j, \quad i = 2,3,4,\ldots \end{aligned} \quad (15)$$

Since $\hat{R}$is a triangular matrix and since every triangular matrix is diagonalizable, provided that no element of its diagonal is zero, every diagonalizable matrix has a square root, then the square root of a triangular matrix is a triangular matrix.

The above allows one to write equations for $\hat{R}$ and $\hat{R}^{1/2}$as

$$\hat{R} = \hat{U}\hat{D}\hat{U}^{-1} \quad (16)$$

$$\hat{R}^{1/2} = \hat{U}\hat{D}^{1/2}\hat{U}^{-1} \quad (17)$$

Where $\hat{U}$is the matrix of change of base of the eigenvectors of $\hat{R}$, $\hat{U}^{-1}$ is the matrix of change of canonical base to the base of eigenvectors of $\hat{R}$ and $\hat{D}$ is the diagonal matrix of eigenvalues of $\hat{R}$.

The calculation of the eigenvalues of $\hat{R}$is carried out by the equation of eigenvalues $(\hat{R} - \mu\hat{I})|\sigma\rangle$. In this way, the matrix $\hat{D}$is defined as follows:

$$\hat{D} = \begin{bmatrix} \mu_1 & 0 & 0 & \cdots & 0 \\ 0 & \mu_2 & 0 & \cdots & 0 \\ 0 & 0 & \mu_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{m+1} \end{bmatrix} \quad (18)$$

Where $\mu_i = r_{i,i}$for $i = 1,2,3,\ldots,m+1$.

The matrix of eigenvectors $\hat{U}$ is:

$$\rho\hat{U} = \begin{bmatrix} k_{1,1}g_1 & k_{1,2}g_2 & k_{1,3}g_3 & \cdots & k_{1,m+1}g_{m+1} \\ 0 & k_{2,2}g_2 & k_{2,3}g_3 & \cdots & k_{2,m+1}g_{m+1} \\ 0 & 0 & k_{3,3}g_3 & \cdots & k_{3,m+1}g_{m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_{m+1,m+1}g_{m+1} \end{bmatrix} \quad (19)$$

Where the $k_{i,j}$are normalization constants defined as:

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$$
\begin{aligned}
&k_{i,i+1} = 1 \quad ; \quad i = 1,2,3,\dots,m \\
&k_{i,i} = (1-\delta_{1,i})\left[\left(\frac{r_{i,i}-r_{i-1,i-1}}{r_{i-1,i}}\right)-1\right]+1 \quad ; \quad i = 1,2,3,\dots,m+1 \\
&k_{i,j} = \frac{1}{r_{j,j}-r_{i,i}}\sum_{s=i+1}^{j} r_{i,s}k_{s,j} \quad ; \quad i \neq j,\ i = 1,2,\dots,m+1;\ j = 2,3,\dots,m+1
\end{aligned} \tag{20}
$$

It is not necessary to calculate the g_i elements since these disappear when the calculation of (16) is performed.

Given that a triangular matrix is invertible if and only if all the elements of the main diagonal are non-zero, the inverse of an upper triangular matrix is another upper triangular matrix. The elements of $\hat{U}^{-1}$ can be calculated through the relation $\hat{U}^{-1}\hat{U} = \hat{I}$. The $u_{i,j}$ elements are:

$$u_{i,j} = \frac{1}{g_i k_{i,i}}\left[1+(\delta_{i,j}-1)\left[1+\sum_{s=i+1}^{j} g_s k_{i,s} u_{s,j}\right]\right] \quad ; \quad i,j = 1,2,3,\dots,m+1 \tag{21}$$

Now, it is possible to calculate equation (17). Taking into account that the product of two triangular matrices is a superior triangular matrix, $\hat{R}^{1/2}$ is a superior triangular matrix with elements $M_{i,j}$ defined as follows:

$$M_{i,j} = \sum_{s=i}^{j} \sqrt{\mu_s} g_s k_{i,s} u_{s,j} \quad ; \quad i,j = 1,2,3,\dots,m+1 \tag{22}$$

In this way, the calculation suggested in equation (13) can be performed; The square root of the variance matrix is:

$$\hat{B}^{1/2} = \begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} & \cdots & B_{1,m+1} \\ B_{2,1} & B_{2,2} & B_{2,3} & \cdots & B_{2,m+1} \\ B_{3,1} & B_{3,2} & B_{3,3} & \cdots & B_{3,m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m+1,1} & B_{m+1,2} & B_{m+1,3} & \cdots & B_{m+1,m+1} \end{bmatrix} \tag{23}$$

Where the elements $B_{i,j}$ are defined by:

$$B_{i,j} = \sum_{s=1}^{i} M_{s,i} M_{s,j} \quad ; \quad i,j = 1,2,3,\dots,m+1 \tag{24}$$

In this way, the square root of the analytical variance matrix is calculated. This method allows for a higher computational efficiency and better numerical approximations of stochastic point kinetics. However, for simplicity in the writing of the elements $B_{i,j}$ it has been decided to approximate these to their dominant term, thus the elements $B_{i,j}$ will be:

$$
\begin{aligned}
&B_{1,1} = \sqrt{\xi} \\
&B_{i,i} = \sqrt{r_{i-1} - \frac{a_{i-1}^2}{\xi}} \quad ; \quad i = 1,2,3,\dots,m+1 \\
&B_{1,i} = B_{i,1} = \frac{a_{i-1}}{\sqrt{\xi}} \quad ; \quad i = 1,2,3,\dots,m+1 \\
&B_{i,j} = \frac{b_{i,j} - \frac{a_{i-1}a_{j-1}}{\xi}}{(r_{i-1}r_{j-1})^{1/4} + r_{i-1}^{1/2}} \quad ; \quad i > j,\ i = 2,3,\dots,m+1;\ j = 3,4,\dots,m+1
\end{aligned} \tag{25}
$$

Next, the implicit Milstein scheme is presented, which will give a numerical solution to the stochastic point kinetics.

IMPLICIT MILSTEIN SCHEME

This section presents the Implicit Milstein scheme [11], written in a discreet manner:

$$x_{n+1} = x_n + a_{n+1}h + b_n\Delta\omega + \frac{1}{2} b_n \frac{\partial}{\partial x_n} b_n[(\Delta\omega)^2 - h] \tag{26}$$

where

$$h = t_{n+1} - t_n \tag{27}$$

$$\Delta\omega = \omega_{n+1} - \omega_n \tag{28}$$

Equation (28) represents a Wiener process, with the following characteristics: $\omega_{t=0} = 0$ with probability 1 and $\omega_t - \omega_s \sim \aleph(0, t-s)$ for $0 \le s \le t$, where $\aleph(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2 [12].

Applied to stochastic point kinetics

$$P_{k+1} = P_k + [A_{k+1}P_{k+1} + Q_{k+1}]h + B_k^{1/2}\Delta\omega_k + \frac{1}{2}B_k^{1/2}\frac{d}{dP_k}B_k^{1/2}[(\Delta\omega_k)^2 - h] \tag{29}$$

Resolving for P_{k+1} we get:

$$P_{k+1} = (I - hA_{k+1})^{-1}\left[P_k + Q_{k+1}h + B_k^{1/2}\Delta\omega_k + \frac{1}{2}B_k^{1/2}\frac{d}{dP_k}B_k^{1/2}[(\Delta\omega_k)^2 - h]\right] \tag{30}$$

Where I is the identity matrix. The calculation of the inverse of $I - hA_{k+1}$ has already been presented analytically [8] and its result will be used in this work. Thus, equation (30) turns out to be:

$$P_{k+1} = S_{k+1}\left[P_k + Q_{k+1}h + B_k^{1/2}\Delta\omega_k + \frac{1}{2}B_k^{1/2}\frac{d}{dP_k}B_k^{1/2}[(\Delta\omega_k)^2 - h]\right] \tag{31}$$

Where

$$\hat{S}=\begin{bmatrix} S_{1,1} & S_{1,2} & S_{1,3} & \cdots & S_{1,m+1} \\ S_{2,1} & S_{2,2} & S_{2,3} & \cdots & S_{2,m+1} \\ S_{3,1} & S_{3,2} & S_{3,3} & \cdots & S_{3,m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{m+1,1} & S_{m+1,1} & S_{m+1,1} & \cdots & S_{m+1,m+1} \end{bmatrix} \quad (32)$$

With

$$S_{1,j}=\frac{1}{\zeta_{k+1}}\left[1+\left(\frac{\lambda_{j-1}\Delta}{1+\lambda_{j-1}\Delta}-1\right)\left(1-\delta_{1,j}\right)\right] \quad ; \; j=1,2,3,\dots,m+1$$
$$\zeta_{k+1}=1-\frac{\rho_{k+1}-\beta}{\Lambda}\Delta-\frac{\Delta^2}{\Lambda}\sum_{i=1}^{m}\frac{\lambda_i\beta_i}{1+\lambda_i\Delta} \quad (33)$$
$$S_{i,j}=\frac{1}{1+\lambda_{j-1}\Delta}\left[\frac{\beta_{i-1}\Delta}{\Lambda}S_{1,j}+\delta_{1,j}\right] \quad ; \quad \begin{matrix} i=2,3,4,\dots m+1 \\ j=1,2,3,\dots,m+1 \end{matrix}$$

Furthermore, $B_k^{1/2}$can be expressed as

$$B_k^{1/2}=\begin{bmatrix} B_{1,1}\frac{d}{dn_k}B_{1,1} & B_{1,2}\frac{d}{dn_k}B_{1,2} & B_{1,3}\frac{d}{dn_k}B_{1,3} & \cdots & B_{1,m+1}\frac{d}{dn_k}B_{1,m+1} \\ B_{2,1}\frac{d}{dC_{1\,k}}B_{2,1} & B_{2,2}\frac{d}{dC_{1\,k}}B_{2,2} & B_{2,3}\frac{d}{dC_{1\,k}}B_{2,3} & \cdots & B_{2,m+1}\frac{d}{dC_{1\,k}}B_{2,m+1} \\ B_{3,1}\frac{d}{dC_{2\,k}}B_{3,1} & B_{3,2}\frac{d}{dC_{2\,k}}B_{3,2} & B_{3,3}\frac{d}{dC_{2\,k}}B_{3,3} & \cdots & B_{3,m+1}\frac{d}{dC_{2\,k}}B_{3,m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m+1,1}\frac{d}{dC_{mk}}B_{m+1,1} & B_{m+1,2}\frac{d}{dC_{mk}}B_{m+1,2} & B_{m+1,3}\frac{d}{dC_{mk}}B_{m+1,3} & \cdots & B_{m+1,m+1}\frac{d}{dC_{mk}}B_{m+1,m+1} \end{bmatrix} (34)$$

Equation (31) is the Milstein's implicit scheme applied to stochastic point kinetics through which the results presented in this work will be obtained. However, it is necessary to consider a way to reduce the variance of neutron and precursor populations, which can improve the approximation to the expected values of these populations. This is achieved by considering a new independent Brownian motion that multiplies the terms of the main diagonal of the square root of the variance matrix, in the following manner:

$$B_{1,1}=d\omega(t)\sqrt{\xi}$$
$$B_{i,i}=d\omega(t)\sqrt{r_{i-1}-\frac{a_{i-1}^2}{\xi}} \quad ; \quad i=1,2,3,\dots,m+1 \quad (35)$$

Since stochastic point kinetics is a generalization of point kinetics, one way to decrease the variance values is to attenuate the contribution of the term that differentiates these two formulations of the physical system. This is why the modification given by equation (35) is made, taking into account that these terms have the greatest contribution. Another way of decreasing variance values is by equating the variance matrix to that of expected values, as in [7]. In the next section, the different numerical experiments will be presented, considering different cases for the form of the reactivity, to obtain the values of the neutron population density and the concentration of delayed neutron precursors.

RESULTS AND DISCUSSIONS

Various numerical experiments are presented for different configurations of reactivity, population of precursors, time steps, Brownian motion and initial conditions. The results obtained by the numerical experiments are compared on mean with the deterministic model (DM) of the point kinetics which is calculated by the implicit Euler scheme, since the deterministic formulation does not have standard deviation values in the tables represented by do Not Apply (NA). They will also be compared with respect to the mean and standard deviation with other stochastic schemes reported in the literature, such as: SPCA (Stochastic piecewise Constant Approximation) and MC (Monte Carlo) [1], EM (Euler-Maruyama) and T 1.5 (Taylor 1.5) [2], FSNPK (Fractional stochastic point kinetics equations) [3], SSPK (Simplified Stochastic Point Kinetics Equations) [4], AEM (Analytical Exponential Model) [5], Double DDM (Double Diagonalization-Decomposition Method) [6], ESM (Efficient Stochastic Model) [7], IEM (Implicit Euler-Maruyama) [8]. The values reported in the literature have been written with four significant figures, in cases where fewer significant figures are reported it has been completed with zeros. In this work, the results in the tables are presented by the acronym IM and IMwDB denoting Implicit Milstein and Implicit Milstein with Diagonal Brownian, respectively.

Experiment 1

This experiment uses a group of precursors and the following physical parameters: reactivity $\rho=-{}^{1}/_{3}\,pcm$, neutron generation time $\Lambda={}^{2}/_{3}\,(s)$, decay constant $\lambda_1=0.1(s^{-1})$, fraction of delayed neutron precursors $\beta=0.05$, average number of neutrons generated by each fission event $\upsilon=2.5$, external source of

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neutrons $q(t) = 200$, initial condition $n(0) = 400$, $C(0) = 300$. This example is done with 40 iterations in a time of [0,2]s, using 5000 Wiener processes. The results obtained for this experiment are shown in Table-1, which clearly shows how the IM scheme achieves good approximations for the expected values of the neutron and precursor populations, the standard deviation values are found in accordance with those reported by other schemes. The IMwDB scheme achieves better approximations for expected values and a considerable decrease in standard deviation values. Figure-1 clearly shows how the use of the proposed method given by equation (35) reduces stochastic variations with respect to the deterministic value, this produces a better approximation of the expected values of the random variables.

The following three experiments consider six groups of precursors with the following physical parameters: neutron generation time $\Lambda = 0.00002(s)$, decay constants $\lambda_i = [0.0127, 0.0317, 0.1150, 0.3110\ 1.4000, 3.8700](s^{-1})$, total fraction of delayed neutron precursors $\beta = 0.007$, fraction of delayed neutron precursors of the *i-th* group $\beta_i = [0.000266, 0.001491, 0.001316, 0.002849, 0.000896, 0.000182]$, average of neutrons generated by fission event $\upsilon = 2.5$, external source of neutrons $q(t) = 0$, initial condition $[n(0), C_1(t), C_2(t), \dots, C_6(t)]^T = 100\left[1, \frac{\beta_1}{\lambda_1\Lambda}, \frac{\beta_2}{\lambda_2\Lambda}, \dots, \frac{\beta_6}{\lambda_6\Lambda}\right]^T$ and using 5000 Wiener processes. Some of these examples represent reactivities for real nuclear reactors.

Table-1. Comparison of results, proposed scheme, reported in the literature and deterministic model for the first experiment.

Método	$\boldsymbol{E[n(2seg)]}$	$\boldsymbol{\sigma[n(2seg)]}$	$\boldsymbol{E[C(2seg)]}$	$\boldsymbol{\sigma[C(2seg)]}$
SPCA	395.3200	29.4110	300.6700	8.3564
MC	400.0300	27.3110	300.0000	7.8073
EM	412.2300	34.3910	315.9600	8.2656
T 1.5	412.1000	34.5190	315.9300	8.3158
AEM	396.2800	31.2120	300.4200	7.9576
Double DDM	402.3500	28.6100	305.8400	7.9240
ESM	396.6200	0.9199	300.3900	0.0016
IEM	399.7100	31.4310	299.7700	7.9411
IEM*	399.9874	0.5439	299.8730	6.8405
IM	400.3121	31.4163	299.8705	8.0535
IMwDB	400.0603	6.8745	299.9968	1.7769
DM	400.0000	NA	300.0000	NA

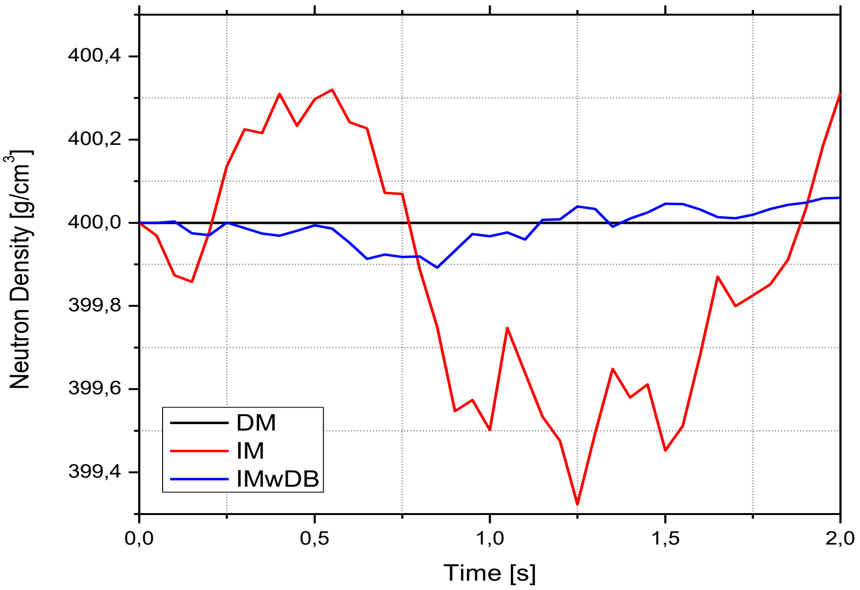

Figure-1. Neutron density in the stochastic model for the IM and IMwDB schemes and the deterministic model (DM) for reactivity $\rho = 300\ pcm$.

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Experiments 2 and 3. are run for six groups of precursors with subcritical and critical reactivities of $\rho = 300\,pcm$ and $\rho = 700\,pcm$, respectively, with a time interval in which the nuclear reactor is simulated with subcritical and critical reactivity of [0,0.1] and [0,0.001] s, respectively. Tables 2 and 3 show the results obtained for these two experiments. The greater accuracy of IMwDB is highlighted on all reported schemes for both expected values and standard deviation. Figure-2 illustrates how the IMwDB scheme manages to approximate with such precision that the IMwDB curve overlaps the DM curve.

Table-2. Comparison of results, proposed scheme, reported in the literature and deterministic model for the second experiment.

Método	$\boldsymbol{E}[\boldsymbol{n}(0.1seg)]$	$\boldsymbol{\sigma}[\boldsymbol{n}(0.1seg)]$	$\boldsymbol{E}[\boldsymbol{C}(0.1seg)]$	$\boldsymbol{\sigma}[\boldsymbol{C}(0.1seg)]$
SPCA	186.3100	164.1600	4.4910×10^5	1.9172×10^3
MC	183.0400	168.7900	4.4780×10^5	1.4957×10^3
EM	208.6000	255.9500	4.4980×10^5	1.2333×10^3
T 1.5	199.4080	168.5470	4.4970×10^5	1.2188×10^3
SSPK	184.8000	186.9600	4.4890×10^5	0.9826×10^3
AEM	186.3000	164.1400	4.4900×10^5	1.9119×10^3
Double DDM	187.0500	167.8300	4.4880×10^5	1.4756×10^3
ESM	179.9300	10.5550	4.4890×10^5	0.0947×10^3
IEM	178.2700	165.1100	4.4886×10^5	1.2536×10^3
IEM*	179.9461	0.2178	4.4888×10^5	0.0604×10^3
IM	180.3313	171.7072	4.4881×10^5	2.0415×10^3
IMwDB	180.0186	8.6228	4.4888×10^5	0.1004×10^3
DM	179.9485	NA	4.4888×10^5	NA

Table-3. Comparison of results, proposed scheme, reported in the literature and deterministic model for the third experiment.

Método	$\boldsymbol{E}[\boldsymbol{n}(0.001seg)]$	$\boldsymbol{\sigma}[\boldsymbol{n}(0.001seg)]$	$\boldsymbol{E}[\boldsymbol{C}(0.001seg)]$	$\boldsymbol{\sigma}[\boldsymbol{C}(0.001seg)]$
SPCA	134.5500	91.2420	4.4640×10^5	19.4440
MC	135.6700	93.3760	4.4640×10^5	16.2260
EM	139.5680	92.0420	4.4630×10^5	6.0710
T 1.5	139.5690	92.0470	4.4630×10^5	18.3370
AEM	134.5400	91.2340	4.4640×10^5	19.2350
Double DDM	135.8600	93.2100	4.4630×10^5	17.8450
ESM	134.9600	6.8527	4.4640×10^5	2.5290
IEM	134.0200	93.2730	4.4636×10^5	18.7760
IEM*	134.9218	5.9661	4.4636×10^5	6.0686
IM	136.0061	93.5743	4.4636×10^5	19.0110
IMwDB	135.0031	0.4736	4.4636×10^5	0.3028
DM	135.0010	NA	4.4636×10^5	NA

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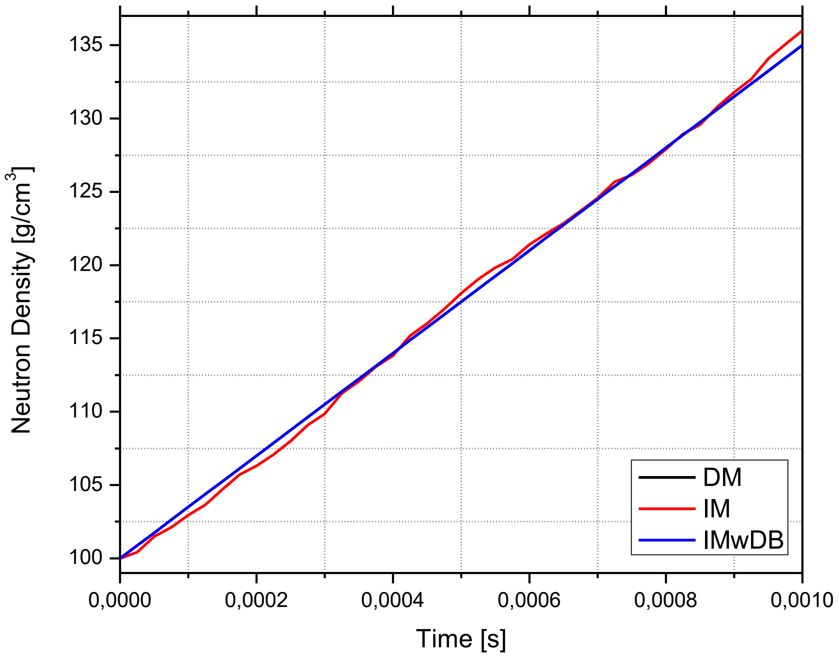

Figure-2. Neutron density in the stochastic model for the IM and ImwDB schemes and the deterministic model (DM) for reactivity $\rho = 0.1\beta tpcm$.

Experiment 4

This uses a linear reactivity of the form $\rho = 0.1\beta t$in a time interval of [0,1]s and using 100 iterations. The results of this numerical experiment are in accordance with those reported in the literature, as can be seen in Table-4. The results improve and are of very good precision for the calculation of the expected value of the neutron density when using the IMwDB method, in addition, the proposed method decreases the variance values when only the IM method is used, obtaining lower values than the SPCA, AEM and IEM methods.

In this section, several experiments were carried out in which the different configurations of the reactivity parameters, precursor population, time steps, Brownian motion and initial conditions were tested, in order to validate the data, found in this study, such as the analytical calculation of the square root of the variance matrix and the correct functioning of the proposed schemes to give a numerical solution to the stochastic point kinetics. The results of these schemes are consistent with the values reported in the literature, and in some cases with better accuracy.

Table-4. Comparison of results, proposed scheme, reported in the literature and deterministic model for the fourth experiment.

Método	$\boldsymbol{E}[\boldsymbol{n}(1seg)]$	$\boldsymbol{\sigma}[\boldsymbol{n}(1seg)]$	$\boldsymbol{E}[\boldsymbol{C}(1seg)]$	$\boldsymbol{\sigma}[\boldsymbol{C}(1seg)]$
SPCA	113.2680	13.3301	$4.4823\text{x}10^5$	$3.0099\text{x}10^3$
AEM	113.2677	13.3272	$4.4823\text{x}10^5$	$3.0026\text{x}10^3$
ESM	113.1164	4.1111	$4.4825\text{x}10^5$	$0.0472\text{x}10^3$
IEM	112.0506	71.3802	$4.4826\text{x}10^5$	$3.0783\text{x}10^3$
IEM*	113.0926	0.2770	$4.4825\text{x}10^5$	$0.1912\text{x}10^3$
IM	112.4026	97.6061	$4.4790\text{x}10^5$	$3.9871\text{x}10^3$
IMwDB	113.3059	10.7728	$4.4827\text{x}10^5$	$1.3718\text{x}10^3$
DM	113.0990	NA	$4.4825\text{x}10^5$	NA

CONCLUSIONS

In this work, the square root of the variance matrix of stochastic point kinetics has been calculated analytically, using the Cholesky decomposition method. A new equation of stochastic point kinetics was obtained by considering an approximation of the dominant terms in some elements of the variance matrix. In order to validate the proposed method, different numerical experiments of the physical system of study were carried out by implementing the implicit Milstein scheme, numerical approximations were found according to those reported in the literature for different forms of nuclear reactivity.

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