# ACCELERATED CONVERGENCE OF NUMERICAL SOLUTION TO SQUARE PLATE BENDING PROBLEM 

Mikhail Ivanovich Popov ${ }^{1}$, Aleksey Vasilyevich Skrypnikov ${ }^{1}$, Vyacheslav Gennadievich Kozlov ${ }^{2}$, Dmitry Gennadievich Kozlov ${ }^{2}$, Petr Viktorovich Tikhomirov ${ }^{3}$, Vadim Yurievich Bukreev ${ }^{2}$, Maksim Aleksandrovich Abasov ${ }^{4}$ and Victor Anatolyevich Khvostov ${ }^{1}$<br>${ }^{1}$ Voronezh State University of Engineering Technologies, Voronezh, Russian Federation<br>${ }^{2}$ Voronezh State Agrarian University named after Emperor Peter the Great, Voronezh, Russia<br>${ }^{3}$ Bryansk State University of Engineering and Technology, Bryansk, Russian Federation<br>${ }^{4}$ Military Training and Research Center of the Air Force "Air Force Academy named after Professor N. E. Zhukovsky and Y. A. Gagarin», Voronezh, Russia<br>E-Mail: mikhail_iv_popov@mail.ru


#### Abstract

The paper presents a numerical solution to the problem of bending a square plate clamped along periphery by the classical finite difference method. The obtained solution is improved using the Richardson extrapolation method, which allowed to reduce the absolute error of the solution by 4 orders of magnitude. A step-by-step construction of an implicit finite difference scheme of the second order of accuracy for an inhomogeneous biharmonic equation under zero boundary conditions on the desired function and its gradient is presented. The classical and modified Richardson extrapolation methods are considered on the sequence of grid functions. The extrapolated solutions are presented as a linear combination of grid functions in common nodes with weight coefficients. Computational experiments for the test function were carried out using the Maple computer environment. Solutions of the problem for different grid steps are obtained. The dependence of the calculation time error on the number of significant digits is revealed. The optimal number of significant digits is found. The rate of convergence of the classical finite difference method is investigated. The analysis of convergence acceleration using the classical and modified Richardson extrapolation method is carried out.


Keywords: square plate, approximation, convergence, bending, numerical solution, error.

## INTRODUCTION

Thin rectangular plates made of various materials are applied upon erection of buildings and industrial facilities, for instance, as reinforced concrete slabs and panels, metal decking, tank bottoms. They are applied in engineering industry as structural elements of various instruments and mechanisms. Rectangular plate bending problems under various supporting conditions along edges appear in the theory of plates and shells (Zav'yalov, Martynov and Romanovskij 2012; Suhoterin, Baryshnikov and Lomteva, 2016). Mathematical simulation of this type of problems leads to biharmonic equation with various boundary conditions. It should be mentioned that in some works, analytical solutions have been obtained for certain formulations in the form of series. The first solution for free supported plate was presented by Navier in 1820 in the form of double trigonometric series. However, the rate of convergence of the obtained series and its derivatives leaves something to be desired. Solution for rectangular plate with two opposite supported edges and two other edges fixed anyhow was obtained by Maurice Lévy. It is comprised of simple series with trigonometric and hyperbolic functions. The rate of convergence is higher than that in the Navier solution but insufficient for accurate computations. Subsequently solutions in the form of series were obtained for other formulations. Poor convergence of trigonometric series requires for accounting for high number of series terms for acceptable accuracy. Since the series coefficients contain trigonometric and hyperbolic functions, the computations should have thousands of decimal positions. These two
circumstances significantly increase computation time. Therefore, both searching for new and improvement of existing numerical methods are urgent issues. The most powerful and universal tool to improve solution accuracy is the Richardson extrapolation. It allows to achieve high accuracy by application of simplest difference approximations of differential operators. Nowadays Richardson extrapolation and its modifications are widely applied in various problems for improvement of solution accuracy (Prihodovskij 2016; Tihovskaya 2015; Korpusov, Luk'yanenko, Ovsyannikov and Panin 2017). The theory of this method is developed for common differential equations and their sets. General theoretical substantiation for equations in partial derivatives is unavailable at present. This work discusses application of this method for solution to square plate bending problem in the case of clamping along overall periphery.

## MATERIALS AND METHODS

Thin square plate with a side equalling to 1 is considered (Figure-1). The plate is clamped along overall periphery, the force $f(x, y)$ is applied to each internal point.
www.arpnjournals.com


Figure-1. Design model.
Mathematical description of bending $u(x, y)$ is a heterogeneous biharmonic equation with boundary conditions of the first and the second orders in square plate region $\bar{D}=[0 ; 1] \times[0 ; 1]$ (Ryazhskih, Slyusarev, Popov, 2013)

$$
\begin{align*}
& \frac{\partial^{4} u}{\partial x^{4}}+2 \frac{\partial^{4} u}{\partial x^{2} \partial y^{2}}+\frac{\partial^{4} u}{\partial y^{4}}=f(x, y),  \tag{1}\\
& u(0, y)=u(1, y)=u(x, 0)=u(x, 1)=0,  \tag{2}\\
& \frac{\partial u(0, y)}{\partial x}=\frac{\partial u(1, y)}{\partial x}=\frac{\partial u(x, 0)}{\partial y}=\frac{\partial u(x, 1)}{\partial y}=0 \tag{3}
\end{align*}
$$

In order to obtain finite-difference scheme, let us replace the solution region $\bar{D}$ with grid with the steps $\Delta x, \Delta y, \quad \Omega_{\Delta x, \Delta y}=\left\{\left(x_{i}, y_{j}\right)=(i \Delta x, j \Delta y)\right\}, i=0, . ., n, j=$ $0, . ., m$. Instead of the function of analog argument in $\bar{D}$, let us consider the function of discrete argument $w\left(x_{i}, y_{j}\right)_{\Delta x, \Delta y}$ denoted as $w_{\Delta x, \Delta y}$, and its values in each node as $w_{i, j}$. Partition of the region by grid is performed so that the boundary points and the center are in the partition nodes.

In order to construct finite-difference analog of partial derivatives $\partial^{4} u / \partial \mathrm{x}^{4}$ and $\partial^{4} \mathrm{u} / \partial \mathrm{y}^{4}$, let us use the central-difference operator of the 2 nd order applied twice in terms of respective variable:

$$
\begin{align*}
& \left.\frac{\partial^{4} u}{\partial x^{4}}\right|_{i, j}=\frac{w_{i+2, j}-4 w_{i+1, j}+6 w_{i, j}-4 w_{i-1, j}+w_{i-2, j}}{\Delta x^{4}}  \tag{4}\\
& \left.\frac{\partial^{4} u}{\partial y^{4}}\right|_{i, j}=\frac{w_{i, j+2}-4 w_{i, j+1}+6 w_{i, j}-4 w_{i, j-1}+w_{i, j-2}}{\Delta y^{4}} \tag{5}
\end{align*}
$$

The finite-difference analog of mixed derivative is:

$$
\begin{align*}
& \left.\frac{\partial^{4} u}{\partial x^{2} \partial y^{2}}\right|_{i, j}=\frac{1}{144 \Delta x^{2} \Delta y^{2}}\left[\left(w_{i+2, j+2}-16 w_{i+2, j+1}+\right.\right. \\
& \left.+30 \mathrm{w}_{\mathrm{i}+2, \mathrm{j}}-16 \mathrm{w}_{\mathrm{i}+2, \mathrm{j}-1}+\mathrm{w}_{\mathrm{i}+2, \mathrm{j}-2}\right) \\
& -16\left(w_{i+1, j+2}-16 w_{i+1, j+1}\right. \\
& \left.+30 w_{i+1, j}-16 w_{i+1, j-1}+w_{i+1, j-2}\right) \\
& +30\left(w_{i, j+2}-16 w_{i, j+1}+30 w_{i, j}\right. \\
& \left.-16 \mathrm{w}_{\mathrm{i}, \mathrm{j}-1}+\mathrm{w}_{\mathrm{i}, \mathrm{j}-2}\right) \\
& -16\left(w_{\mathrm{i}-1, \mathrm{j}+2}-16 \mathrm{w}_{\mathrm{i}-1, \mathrm{j}-2}\right. \\
& \left.+30 w_{i-1, j}-16 w_{i-1, j-1}+w_{i-1, j-2}\right)+\left(w_{i-2, j+2}\right. \\
& -16 w_{i-2, j+1}+ \\
& \left.\left.+30 w_{i-2, j}-16 w_{i-2, j-1}+w_{i-2, j-2}\right)\right] \tag{6}
\end{align*}
$$

Substituting equations (4)-(6) into equation. (1), we obtain the difference equation:
$\frac{w_{i+2, j}-4 w_{i+1, j}+6 w_{i, j}-4 w_{i-1, j}+w_{i-2, j}}{\Delta x^{4}}+$
$\frac{w_{i, j+2}-4 w_{i, j+1}+6 w_{i, j}-4 w_{i, j-1}+w_{i, j-2}}{\Delta y^{4}}+\frac{1}{72 \Delta x^{2} \Delta y^{2}}\left[\left(w_{i+2, j+2}-\right.\right.$
$\left.-16 w_{i+2, j+1}+30 w_{i+2, j}--16 w_{i+2, j-1}+w_{i+2, j-2}\right)-$
$16\left(w_{i+1, j+2}-16 w_{i+1, j+1}++30 w_{i+1, j}-16 w_{i+1, j-1}+\right.$
$\left.w_{i+1, j-2}\right)+30\left(w_{i, j+2}-16 w_{i, j+1}+30 w_{i, j}-16 w_{i, j-1}+\right.$
$\left.w_{i, j-2}\right)-16\left(w_{i-1, j+2}-16 w_{i-1, j-2}+30 w_{i-1, j}-\right.$
$\left.16 w_{i-1, j-1}+w_{i-1, j-2}\right)+\left(w_{i-2, j+2}-\right.$
$\left.\left.16 w_{i-2, j+1}+30 w_{i-2, j}-16 w_{i-2, j-1}+w_{i-2, j-2}\right)\right]=f_{i, j}$
The boundary conditions, equation (2), applied to the required function will be as follows:
$w_{0, j}=w_{n, j}=w_{i, 0}=w_{i, n}=0, i=\overline{0, n}, j=\overline{0, n}$.
The boundary conditions applied to the function gradient will be obtained using difference operators for the left and the right boundaries:
$\left.\frac{\partial u}{\partial x}\right|_{i, 0}=\frac{-3 w_{i, 0}+4 w_{i, 1}-w_{i, 2}}{2 \Delta x}, i=1, \ldots, n-1$
$\left.\frac{\partial u}{\partial x}\right|_{i, n}=\frac{3 w_{i, n}-4 w_{i, n-1}+w_{i, n-2}}{2 \Delta x}, i=1, \ldots, n-1$
For the upper and the lower boundaries, the following is valid, respectively:
$\left.\frac{\partial u}{\partial y}\right|_{0, j}=\frac{-3 w_{0, j}+4 w_{1, \mathrm{j}}-w_{2, \mathrm{j}}}{2 \Delta y}, j=1, \ldots, n-1$
$\left.\frac{\partial u}{\partial y}\right|_{n, j}=\frac{3 w_{n, j}-4 w_{n-1, j}+w_{n-2, j}}{2 \Delta y}, j=1, \ldots, n-1$
Let us substitute the derivatives in equation (3) with their finite-difference analogs, equations (9)-(12). Then, taking into account equation (8), the boundary conditions for the derivatives will be as follows:
www.arpnjournals.com
$w_{i, 1}=\frac{1}{4} w_{i, 2}, w_{i, n-1}=\frac{1}{4} w_{i, n-2}, i=1, \ldots, n-1$
$w_{1, j}=\frac{1}{4} w_{2, j}, w_{n-1, j}=\frac{1}{4} w_{n-2, j}, j=1, \ldots, n-1($
Therefore, the boundary problem, equations (1)(3), is replaced with the finite-difference scheme, equations (7), (8), (13), (14). This scheme is implicit.

## RESULTS

Let us denote in uniform grid $\Delta X=\Delta Y=h$ :

$$
B_{h}=\left.\frac{\partial^{4} \Phi}{\partial X^{4}}\right|_{i, j}+\left.2 \frac{\partial^{4} \Phi}{\partial X^{2} \partial Y^{2}}\right|_{i, j}+\left.\frac{\partial^{4} \Phi}{\partial Y^{4}}\right|_{i, j}
$$

Then, equation (7) is as follows:

$$
B_{h} \Phi=-1
$$

In order to determine the approximation error, let us form the difference $z=u-v$, where $u$ is the problem solution, equation (7), (8), (13), (14), and $v$ is the problem solution, equation (1)-(3). Substituting $u=z+v$ into the equation, we obtain the problem for $z$ :
$B u=B(z+v)=B z+B v=-1, \quad B z=-\varepsilon_{B}$
in internal grid nodes and for $z$, the boundary conditions are valid: equations (8), (13), (14), where $\varepsilon_{B}=B_{h} v+1$ is the approximation error of equations (1)-(3) by equations (7), (8), (13), (14). Since $\Delta^{2} v+1=0$, then:
$\varepsilon_{B}=B v+1-\Delta^{2} v-1=B v-\Delta^{2} v$.
Let us calculate the approximation error term by term using the Taylor expansion with remainder term in Lagrange form in the vicinity of the $i, j$-th node:
$\left.\frac{\partial^{4} v}{\partial X^{4}}\right|_{i, j}-\frac{\partial^{4} v(X, Y)}{\partial X^{4}}=\frac{h^{2}}{6} \frac{\partial^{6} v}{\partial X^{6}}[h(i+\eta), j h], \quad 0<\eta<1$
$\left.\frac{\partial^{4} v}{\partial Y^{4}}\right|_{i, j}-\frac{\partial^{4} v(X, Y)}{\partial Y^{4}}=\frac{h^{2}}{6} \frac{\partial^{6} v}{\partial y^{6}}[i h, h(j+\sigma)], \quad 0<\sigma<1$
$\left.\frac{\partial^{4} v}{\partial X^{2} \partial Y^{2}}\right|_{i, j}-\frac{\partial^{4} v(X, Y)}{\partial X^{2} \partial Y^{2}}=\frac{17 h^{4}}{756}\left\{\frac{\partial^{8} v}{\partial Y^{8}}[(i+\eta) h, h j]+\frac{\partial^{8} v}{\partial Y^{8}}[i h, h(j+\sigma)]\right\}-$
$-\frac{34 h^{4}}{189}\left\{\frac{\partial^{8} v}{\partial X^{7} \partial Y}[(i+\eta) h, h(j+\sigma)]+\frac{\partial^{8} v}{\partial Y^{7} \partial X}[(i+\eta) h, h(j+\sigma)]\right\}-$
$-\frac{167 h^{4}}{270}\left\{\frac{\partial^{8} v}{\partial X^{6} \partial Y^{2}}[(i+\eta) h, h(j+\sigma)]+\frac{\partial^{8} v}{\partial Y^{6} \partial X^{2}}[(i+\eta) h, h(j+\sigma)]\right\}-$
$-\frac{34 h^{4}}{27}\left\{\frac{\partial^{8} v}{\partial X^{5} \partial Y^{3}}[(i+\eta) h, h(j+\sigma), k \tau]+\frac{\partial^{8} v}{\partial Y^{5} \partial X^{3}}[(i+\eta) h, h(j+\sigma), k \tau]\right\}+$
$+\frac{85 h^{4}}{54} \frac{\partial^{8} v}{\partial X^{4} \partial Y^{4}}[(i+\eta) h, h(j+\sigma)], 0<\eta<1,0<\sigma<1$.

Let us denote:
$M_{1}=\max _{(X, Y) \in \bar{D}}\left|\frac{\partial^{6} v(X, Y)}{\partial X^{6}}\right|, \quad M_{2}=\max _{(X, Y) \in \bar{D}}\left|\frac{\partial^{6} v(X, Y)}{\partial Y^{6}}\right|$,
$M_{3}=\min _{(X, Y) \in \bar{D}}\left|\frac{\partial^{m+n} v(X, Y)}{\partial X^{m} \partial Y^{n}}\right|, m, n=1,3,5,7, m+n=8 ;$
$M_{4}=\max _{(X, Y) \in \bar{D}}\left|\frac{\partial^{m+n} v(X, Y)}{\partial X^{m} \partial Y^{n}}\right|, m, n=2,4,6,8, m+n=8$.
Since the biharmonic operator is symmetric, $M_{1}=M_{2}$. Let us denote $\tilde{M}_{1}=M_{1} / 3$, $\tilde{M}_{2}=(5398 / 945) M_{4}-(1088 / 189) M_{3}$, then:
$\left|\varepsilon_{B}\right| \leq \tilde{M}_{1} h^{2}+\tilde{M}_{2} h^{4}$.
Therefore, the difference operator approximates the biharmonic term with the 2 nd order.

The boundary condition (2) for the required function with consideration for equation (8) is approximated exactly. Let us estimate the approximation error of equation (3) by equations (13) and (14) using the Taylor expansion with remainder term in Lagrange form in the vicinity of the $i, j$-th node:

$$
\left.\frac{\partial v}{\partial X}\right|_{i, j}-\frac{\partial v}{\partial X}=\frac{2}{3} h^{2} \frac{\partial^{3} v}{\partial X^{3}}[h(i+\eta), j h], \quad 0<\eta<1
$$

Let us denote:
$M=\max _{(X, Y) \in \bar{D}}\left|\frac{\partial^{3} v(X, Y)}{\partial X^{3}}\right|$,
then: $\left|\varepsilon_{\Gamma}\right| \leq M h^{2}$.
Therefore, the boundary condition (2) is approximated also with the second order, hence, the finitedifference scheme, equations (7), (8), (13), (14), approximates the boundary problem, equations (1)-(3), with the accuracy of the second order | (Popov, 2018).

## CLASSICAL SOLUTION ANALYSIS

Let us introduce the test function $f(x, y)$ with the accurate solution to equations (1)-(3). With this aim, let us take the function $u(x, y)$ satisfying the boundary conditions (2)-(3). Let $u(x, y)=\frac{1}{8 \pi^{4}} \sin ^{2}(\pi x) \sin ^{2}(\pi y)$. Let us verify fulfillment of the boundary conditions: $u(0, y)=\frac{1}{8 \pi^{4}} \sin ^{2}(\pi \cdot 0) \sin ^{2}(\pi y)=0, \quad u(1, y)=$ $\frac{1}{8 \pi^{4}} \sin ^{2}(\pi \cdot 1) \sin ^{2}(\pi y)=0, \quad$ similarly $\quad u(x, 0)=$ $u(x, 1)=0$, that is, the boundary conditions (2) are met. $\frac{\partial u}{\partial x}=\frac{1}{8 \pi^{4}} \sin (2 \pi x) \sin ^{2}(\pi y)$, then, $\frac{\partial u(0, y)}{\partial x}=\frac{1}{8 \pi^{4}} \sin (2 \pi$.
$0) \sin ^{2}(\pi y)=0$ and $\frac{\partial u(1, y)}{\partial x}=\frac{1}{8 \pi^{4}} \sin (2 \pi \cdot 1) \sin ^{2}(\pi y)=$ 0 , similarly $\frac{\partial u(x, 0)}{\partial y}=\frac{\partial u(x, 1)}{\partial y}=0$, hence, equation (3) is also met. Let us substitute $u(x, y)$ into equation (1), then, $f(x, y)=\left(8 \cos ^{2}(\pi y)-5\right) \cos ^{2}(\pi x)-5 \cos ^{2}(\pi y)+3$.

Numeric experiments will be performed in Maple environment. Let us determine optimum number of nonzero digits in the computation experiments. Let us preset eight nonzero digits as initial number. While adding by two nonzero digits, let us monitor the dynamics of absolute error and computation time. Table-1 summarizes step-by-step variation of absolute error and computation time for various partitions. It can be seen in the table that upon transfer from 8 to 10 digits, the error decreases insignificantly, and upon further increase from 10 to 14 , it is so low that the computations are not affected. Computation time to 14 digits also increases insignificantly. However, at 16 nonzero digits the computation time sharply increases with increase in error, though, insignificant. Thus, the optimum number is 10 nonzero digits, it is used in Maple by default.

Let us analyze the rate of convergence of finitedifference scheme. We will increase the number of grid nodes by about two times and monitor the variations of computation error, computation time, and consumed memory. Table- 3 summarizes the experimental results. It follows from the data that upon increase in grid nodes by two times, the error decreases by about two times. On this basis, let us derive empirical expression of grid number as a function of absolute error $\varepsilon$.
$h=\frac{125 \sqrt{\varepsilon}}{8 \sqrt{1.37}}$
Let us calculate the solution to equations (7)-(8), (13)-(14) in uniform grid with the step $h=1 / 180$. The obtained solution is illustrated in Figure-2. The solution is symmetric with regard to the central point $(0.5 ; 0.5)$ where the maximum is reached equating to $1.28 \cdot 10^{-3}$. Due to the symmetry, the solution behavior is illustrated informatively by its profile in the cross sections by planes $\mathrm{X}=0.5$ and $\mathrm{Y}=0.5$ in Figure-3. Figure-4 illustrates absolute error of the solution in cross section by plane $\mathrm{X}=0.5$. It can be seen that the maximum absolute error is reached in the central point equating to $1.87 \cdot 10^{-7}$. Computation of solution with higher accuracy is accompanied by two issues. The first issue is increase in computation time. It follows from Table-2 that the step decrease from $1 / 128$ to $1 / 180$ leads to tenfold increase in time. The second issue is increase in required RAM. It follows from Table 2 that the step decrease from $1 / 128$ to $1 / 180$ leads to twofold increase in RAM. And, while it is possible to accept the first issue, then the second issue results in impossibility to solve the problem using the available equipment. Thus, it is necessary to search for other approaches to achieve the required accuracy.

An approach to develop classical finite-difference method improving the rate of convergence is the Richardson extrapolation and its modifications. Let us describe briefly this approach.

In the region $\bar{D}$ let us consider uniform grid $\Omega_{h}=\left\{\left(x_{i}, y_{j}\right)=(i h, j h)\right\}, i=0, . ., n, j=0, . ., n \quad$ with equal steps of variables $\Delta x=\Delta y=h$. Let us plot the sequence of grids $\Omega_{h}^{k}=\left\{\left(x_{i}, y_{j}\right)=\left(i h_{k}, j h_{k}\right)\right\}, i=$ $0, . ., n k, j=0, . ., n k, k=1, \ldots, p$. It is obvious that each grid of the sequence contains the nodes of the first grid. Let us calculate the grid function $w_{h}^{k}$ for each grid of the sequence. The extrapolated solution will be presented in the form of linear combination of grid functions $w_{h}^{k}$ in common nodes $W_{h}^{p}=\sum_{k=1}^{p} a_{k}^{p} w_{h}^{k}$. In the classical method, the weighting factors $a_{k}^{p}$ are obtained from the set of linear equations:
$\sum_{k=1}^{p} a_{k}^{p}=1, \sum_{k=1}^{p} a_{k}^{p} h_{k}^{j}=0, j=1, \ldots, p-1$
Let us select the parameter $h_{k}=\frac{h}{k}$ for the grid sequence $\Omega_{h}^{k}$, then equation (16) is rewritten as follows:
$\sum_{k=1}^{p} a_{k}^{p}=1, \sum_{k=1}^{p} \frac{a_{k}^{p}}{k^{j}}=0, j=1, \ldots, p-1$.
While solving this set, we obtain the equation of weighted factors:
$\mathrm{a}_{\mathrm{k}}^{\mathrm{p}}=\frac{(-1)^{\mathrm{p}-\mathrm{k}} \mathrm{k}^{\mathrm{p}}}{\mathrm{k}!(\mathrm{p}-\mathrm{k})!}$.
The weighted factors for the grid sequences $2-5$ are summarized in Table-3. It can be seen that the weights $a_{k}^{p}$ increase significantly with the number of grid functions. This trend increases the influence of rounding errors, which leads to increase in the number of decimal positions and, as a consequence, the computation time.

Contrary to the classical Richardson method, we will use expansion only in even degrees $h_{k}^{2}$, then:
$\sum_{\mathrm{k}=1}^{\mathrm{p}} \mathrm{b}_{\mathrm{k}}^{\mathrm{p}}=1, \sum_{\mathrm{k}=1}^{\mathrm{p}} \mathrm{b}_{\mathrm{k}}^{\mathrm{p}} \mathrm{h}_{\mathrm{k}}^{2 \mathrm{j}}=0, \mathrm{j}=1, \ldots, \mathrm{p}-1$.
The rate of convergence for common differential equations in this approach is higher than in the classical variant. Let us reduce the parameter $h_{k}$ by $h_{k}=h / k$, then, equation (17) is reduced as follows:
$\sum_{k=1}^{p} b_{k}^{p}=1, \sum_{k=1}^{p} \frac{b_{k}^{p}}{k^{2 j}}=0, j=1, \ldots, p-1$.
While solving this set, we will obtain the equations of weighting factors:
$b_{k}^{p}=2 \frac{(-1)^{p-k} k^{2 p}}{(p+k)!(p-k)!}$.
Table-4 summarizes the weighting factors calculated by this equation for various $p$. The table demonstrates more moderate increase in the weights
$b_{k}^{p}$ upon increase in the number of grid functions, which decreases the influence of rounding errors in comparison with the classical method.

It follows from the table that the higher is the number of grid functions used for improved solution, the higher is the contribution of the last function $w_{h}^{p}$ and the lower is the contribution of the first function $w_{h}^{1}$. We obtain the extrapolated solution only in the nodes of initial grid $\Omega_{h}$, hence, it should contain as many nodes as possible. However, increase in the number of nodes of initial grid leads to significant (by about a factor of $p^{2}$ ) increase in the number of nodes of final grid. Two issues arise in this regard. Firstly, how many grid functions should be combined to obtain the maximum accuracy. Secondly, how to calculate the function in intermediate nodes. The answers will be obtained on the basis of numerical experiments.

## DISCUSSIONS

In order to answer the question formulated above, let us perform the following experiment. Since the grid dimension is constrained by PC memory and computation time, let us take the maximum partition with the step $h=1 / 180$. Let us increase the grid step by a factor of 2 , 3 , etc. Table- 5 summarizes the absolute error as a function of number of grid functions in the extrapolated solution obtained by the classical method, Table-6 shows the same for the modified Richardson method. The column titled Relative error reduction shows the percent of error decrease in comparison with previous result. The first line shows the difference from the classical solution at $h=1 / 180$. It can be seen that combination of two grid functions in the classical method impairs the result and in the modified method the accuracy is improved by an order of magnitude.

While comparing the data in the tables, it can be seen that the modified method is more efficient for two and three grid functions, in the case of four and more functions, the classical method provides better result. However, with increase in the number of decimal positions, the error of the classical method increases. For instance, with increase in the number of nonzero digits from 10 to 15 for the grid function $W_{1 / 30}^{6}$, the absolute error increases by $50 \%$. On the contrary, while using the modified method under the same conditions, the error decreases, though only by $1 \%$. This can be attributed to higher increase in the modulus of weighted factors of the classical method. Figure-5 illustrates the error of solution extrapolated by modified solution $W_{1 / 30}^{6}$.

The rate of error decrease of the modified method rapidly drops, in the case of classical method, this drop is somewhat slower, hence, further increase in the number of grid functions is unreasonable. It should be noted that the number of grid nodes is inversely proportional to the square of term number of the extrapolated solution. Therefore, application of up to five grid functions is optimum for computations.

The function values in intermediate nodes will be calculated by interpolation methods. This will be
exemplified by the extrapolated function $W_{1 / 60}^{3}$. Using the nodes in the central cross section $X=0.5$, let us plot the polynomial spline of the 5th degree. Figure-6 illustrates the spline absolute error. It can be seen that the absolute error in any point does not exceed the value in central point, hence, the spline accuracy coincides with that of the extrapolated function. It is obvious that spline function can be similarly plotted in any cross section. Proceeding this process, it is possible to obtain function value in any point of the region. However, the spline error for $W_{1 / 30}^{6}$ exceeds the error of the function itself. Another method to obtain the values in intermediate nodes is selection of $h_{k}$ as a function of number of grid functions.

As it turned out, the answers for the first and the second questions are interrelated and depend on formulated purposes. If it is required to obtain solution in total region, then three-four terms are sufficient. If we are interested in the solution in a specific point, for instance, in the region center, then it is required to take maximum number of terms. Herewith, time consumption and PC memory should also be taken into account.

In order to estimate the accelerated convergence of the Richardson extrapolation, let us use empiric equation (15). Accuracy of $W_{1 / 60}^{3}$ could be obtained by classical method with the grid step $h=1 / 16000$. This step is lower by a factor of 89 than the step of the extrapolated solution. Computations with such step in Maple environment are impossible using PC.


Figure-2. Solution graph.
www.arpnjournals.com


Figure-3. Solution profile in section $X=0.5$.


Figure-4. Absolute error in section $X=0.5$.


Figure-5. Absolute error of the modified method.


Figure-6. Absolute error of the spline in the section plane

$$
X=0.5 .
$$

## CONCLUSIONS

Using classical finite difference method, the implicit finite-difference scheme has been developed for numerical solution to square plate bending problem. Using expansion of the function into the Taylor series, the approximation error of biharmonic operator has been predicted by difference analog as well as the approximation error of boundary conditions. The rate of convergence of the scheme has been analyzed. Test solutions for various grid steps have been obtained in the form of grid functions. Using the Richardson extrapolation, the extrapolated solution has been obtained in the form of linear combination of grid functions. Classical and modified approaches are analyzed. It has been demonstrated that the numerical algorithm with solution extrapolation is characterized by higher rate of convergence in comparison with the classical approach. The absolute error of solution for this algorithm is by 3-4 orders of magnitude lower than for conventional finitedifference method. Similar accuracy, upon application of
classical method, can be achieved by decreasing grid step by a factor of hundreds, which would increase the computation time by a factor of thousands. Herewith, the requirements to PC RAM would also increase by a factor of hundreds. The function intermediate values are obtained by means of polynomial spline of the fifth degree. In thus approach, the absolute error of the solution is not increased and the computation time varies insignificantly. The obtained results agree with approximate analytical solution for similar problem (Popov, Soboleva, 2016).

Let us mention the advantages of the Richardson extrapolation. Universality: it is suitable both for common differential equations and for equations in partial derivatives. Simplicity of application: simple equations of weighting factors. Efficiency: combination of several solutions provides accuracy by several orders of magnitude higher than accuracy of each single solution.

Its disadvantage, rather serious, is comprised of upscale of computational grid. However, this disadvantage can be compensated by a priori knowledge of solution and application of spline interpolation.

Table-1. Rounding error.

| Number <br> of <br> nonzero <br> digits | Grid step 1/40 |  | Grid step 1/100 |  | Grid step 1/160 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Absolute error | Computation <br> time (s) | Absolute error | Computation <br> time (s) | Absolute error | Computation <br> time (s) |
| 8 | $3.22974 \cdot 10^{-6}$ | 2.7 | $5.86 \cdot 10^{-7}$ | 14 | $2.357 \cdot 10^{-7}$ | 83 |
| 10 | $-0.001 \%$ | $0 \%$ | $-0.002 \%$ | $0 \%$ | -0.04 | $0 \%$ |
| 12 | $-0.00003 \%$ | $0 \%$ | $-0.00008 \%$ | $0 \%$ | $-0.0004 \%$ | $1 \%$ |
| 14 | $-2.17 \cdot 10^{-7} \%$ | $1 \%$ | $-4.1 \cdot 10^{-7} \%$ | $1 \%$ | $-4.24 \cdot 10^{-7} \%$ | $2 \%$ |
| 16 | $5 \cdot 10^{-7} \%$ | $492 \%$ | $0.0002 \%$ | $7,120 \%$ | $0.0003 \%$ | $20,000 \%$ |

Table-2. Rate of convergence.

| Grid <br> step | Absolute error | Relative error <br> reduction (\%) | Computation <br> time (s) | PC memory (MB) |
| :---: | :---: | :---: | :---: | :---: |
| $1 / 32$ | $4.76 \cdot 10^{-6}$ |  | 2.5 | 57 |
| $1 / 44$ | $2.72 \cdot 10^{-6}$ | $43 \%$ | 3 | 66 |
| $1 / 64$ | $1.37 \cdot 10^{-6}$ | $50 \%$ | 4 | 161 |
| $1 / 90$ | $7.17 \cdot 10^{-7}$ | $48 \%$ | 10 | 261 |
| $1 / 128$ | $3.64 \cdot 10^{-7}$ | $49 \%$ | 34 | 974 |
| $1 / 180$ | $1.87 \cdot 10^{-7}$ | $49 \%$ | 418 | 7,564 |

Table-3. Weighting factor $a_{k}^{p}$.

| $\boldsymbol{p} \boldsymbol{k}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | -1 | 2 |  |  |  |  |
| 3 | $\frac{1}{2}$ | -4 | $\frac{9}{2}$ |  |  |  |
| 4 | $-\frac{1}{6}$ | 4 | $-\frac{27}{2}$ | $\frac{32}{3}$ |  |  |
| 5 | $\frac{1}{24}$ | $-\frac{8}{3}$ | $\frac{81}{4}$ | $-\frac{128}{35}$ | $\frac{625}{24}$ |  |
| 6 | $-\frac{1}{120}$ | $\frac{4}{3}$ | $-\frac{81}{4}$ | $\frac{256}{3}$ | $-\frac{3,125}{24}$ | $\frac{324}{5}$ |

Table-4. Weighting factor $b_{k}^{p}$.

| $\boldsymbol{p}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\boldsymbol{k}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ |
| 2 | $-\frac{1}{3}$ | $\frac{4}{3}$ |  |  |  |  |
| 3 | $\frac{1}{24}$ | $-\frac{16}{25}$ | $\frac{81}{40}$ |  |  |  |
| 4 | $-\frac{1}{360}$ | $\frac{16}{45}$ | $-\frac{729}{280}$ | $\frac{1,024}{315}$ |  |  |
| 5 | $\frac{1}{8,640}$ | $-\frac{64}{945}$ | $\frac{6,561}{4,480}$ | $-\frac{16,384}{2,835}$ | $\frac{390,625}{72,576}$ |  |
| 6 | $-\frac{1}{302,400}$ | $\frac{8}{945}$ | $-\frac{2,187}{4,480}$ | $\frac{65,536}{14,175}$ | $-\frac{9,765,625}{798,336}$ | $\frac{17,496}{1,925}$ |

Table-5. Acceleration of convergence of the classical method.

| Number of grid <br> functions $\boldsymbol{p}$ | Step of initial grid | Absolute error | Relative error reduction (\%) |
| :---: | :---: | :---: | :---: |
| 2 | $1 / 90$ | $3.43 \cdot 10^{-7}$ | -93 |
| 3 | $1 / 60$ | $3.55 \cdot 10^{-8}$ | 90 |
| 4 | $1 / 46$ | $1.85 \cdot 10^{-10}$ | 99 |
| 5 | $1 / 36$ | $4.15 \cdot 10^{-11}$ | 76 |
| 6 | $1 / 30$ | $2.22 \cdot 10^{-11}$ | 47 |

Table-6. Accelerating the convergence of the modified method.

| Number of grid <br> functions $\boldsymbol{p}$ | Step of initial grid | Absolute error | Relative error reduction (\%) |
| :---: | :---: | :---: | :---: |
| 2 | $1 / 90$ | $1.049 \cdot 10^{-8}$ | 94 |
| 3 | $1 / 60$ | $3.526 \cdot 10^{-9}$ | 66 |
| 4 | $1 / 46$ | $1.862 \cdot 10^{-9}$ | 47 |
| 5 | $1 / 36$ | $1.325 \cdot 10^{-9}$ | 29 |
| 6 | $1 / 30$ | $9.597 \cdot 10^{-10}$ | 28 |

## REFERENCES

Korpusov M. O., Luk'yanenko D. V., Ovsyannikov E. A. and Panin A. A. 2017. Local solvability and destruction of the solution of one equation with quadratic non-coercive nonlinearity. Mathematical modeling and programming. 2: 107-123.

Popov M. I. and Soboleva E. A. 2016. Approximate analytical solution of the internal problem of conductivelaminar free convection Bulletin of the Voronezh state University of engineering technologies. 4: 78-84.

Popov M. I. 2018. Integration of inhomogeneous biharmonic equation by implicit scheme Bulletin of Voronezh state University of engineering technologies. 2: 114-118.

Prihodovskij M. A. 2016. Modification of Richardson's extrapolation method International journal of applied and fundamental research. 12: 237-243.

Ryazhskih V. I, Slyusarev M. I., Popov M. I. 2013. Numerical integration of the biharmonic equation in the square domain Bulletin of St. Petersburg University. Applied mathematics. Informatics. Management process. 2013: 52-62.

Suhoterin M. V., Baryshnikov S. O. and Lomteva K. O. 2016. On homogeneous solutions to the problem of bending a rectangular cantilever plate Nauchnotekhnicheskie Vedomosti SPbGPU. Physical and mathematical Sciences. 3: 248.

Tihovskaya S. V. 2015. Investigation of the two-grid method of increased accuracy for the elliptic reaction diffusion equation with boundary layers. Scientific notes of Kazan University. 157: 205.

Zav'yalov V. N., Martynov E. A. and Romanovskij V. M. 2012. Basics of structural mechanics of plates: textbook 2 Omsk: SibADI. p. 116.

