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GALERKIN-VLASOV METHOD FOR THE ELASTIC BUCKLING ANALYSIS OF KIRCHHOFF PLATE WITH ONE FREE EDGE AND THREE SIMPLY SUPPORTED EDGES UNDER UNIFORM UNIAXIAL COMPRESSION

Onyedikachi Aloysius Oguaghamba[1] and Charles Chinwuba Ike[2]
[1]Department of Civil Engineering, University of Nigeria, Nsukka, Nigeria
[2]Department of Civil Engineering, Enugu State University of Science and Technology, Enugu State, Nigeria
E-Mail: onyedikachi.oguaghamba@unn.edu.ng

ABSTRACT

This work presents the Galerkin-Vlasov method for solving the elastic buckling problem of Kirchhoff plate (length a and width b) under uniaxial uniform compressive load applied at the two opposite simply supported edges ($x = 0$ and $x = a$) with the edge $y = 0$ simply supported and the edge $y = b$ free. Mathematically, the problem is a boundary value problem (BVP) represented by a partial differential equation (PDE) over the domain subject to boundary conditions at the plate edges. Upon suitable selection of basis functions the Galerkin-Vlasov method converts the domain equation to an integral equation, and ultimately to ordinary differential equations (ODE). The ODE is solved, and boundary conditions along $y = 0$, and $y = b$ for the considered problem used to generate system of homogeneous equations in terms of the integration constants. The characteristic buckling equation is found as a transcendental equation from the condition for nontrivial solutions of the system of homogeneous equations. The roots of the transcendental equation obtained by computational software and iterative techniques are used to obtain the elastic buckling loads for the first two buckling modes, for various aspect ratios (a/b) and for Poisson ratio of $\mu = 0.25$. It is found that the critical elastic buckling load occurs at the first buckling mode, and the values of the critical elastic buckling loads computed are in close agreement with values obtained previously by Timoshenko.

Keywords: galerkin-Vlasov method, elastic buckling problem, Kirchhoff plate, characteristic buckling equation, critical elastic buckling load, elastic buckling load coefficient.

INTRODUCTION

Elastic buckling problems of plates arise when plates are subjected to loads applied in a direction parallel to their plane and are commonly encountered in engineering [1-5]. Usually buckling failures occur suddenly, and may result in catastrophic structural failure. Hence, it is necessary to determine the buckling load capacities of the plates in order to avert premature failure.

Early studies of plate buckling could be traced to Navier who derived using Kirchhoff's assumptions the partial differential equation of stability of rectangular plates. Since then, research studies on the buckling of plates with varieties of shapes, boundary and loading conditions have been reported by Timoshenko and Gere [1], Bulson [2], Gambhir [3], Chajes [4], Wang *et al* [5], Shi [6], Shi and Bezine [7], Abodi [8], Batford and Houbolt [9], Wang *et al* [10], Xiang et al [11], Ullah *et al* [12], Ullah *et al* [13], Abolghasemi *et al* [14].

The buckling of plates may be categorized as elastic buckling and plastic buckling. In elastic buckling problems, it is assumed that the critical buckling load is less than the elastic limit of the plate material. However the plate may be stressed beyond the elastic limit before buckling occurs, and buckling theories of plasticity are needed to describe such plastic (inelastic) buckling problems. Generally two plastic theories - the deformation theory and the incremental theory of plasticity - are used for plastic buckling problems (Yu Chen, [15]).

Navier derived using Kirchhoff's assumptions the governing partial differential equation for the elastic stability of thin rectangular plates under lateral load by including the twisting action. The inclusion of the twisting term was an important contribution to the theory of plates because the resistance of the plate to twisting can greatly reduce the deflections under lateral load.

Saint Venant modified the Navier's governing equation for elastic stability of thin rectangular plates by the inclusion of axial edge forces and shearing forces. Ibearugbulem [16] used the direct variational principle for the analysis of the elastic stability problems of thin rectangular flat plates under various boundary conditions. Oguaghamba [17] analysed the buckling and post-buckling loads of isotropic thin rectangular plates. Oguaghamba *et al* [18] studied the buckling and post-buckling loads characteristics of thin rectangular plate with clamped edges.

Nwadike [19] used the Ritz method for the buckling analysis of isotropic rectangular plates. Ibearugbulem *et al* [20] and Osadebe *et al* [21] used the Taylor-Maclaurin series as shape functions in the Galerkin's variational method for the stability analysis of simply supported thin rectangular plates. Abodi [8] used the finite difference method for the elastic buckling analysis of plates under in-plane patch loading.

Shi [6] and Shi and Bézíne [7] used the boundary element method for the buckling analysis of orthotropic plates. Nwoji *et al* [22] used the double finite Fourier sine integral method for the elastic buckling analysis of simply supported thin rectangular plates under uniaxial uniform compressive loads and obtained exact solutions for the

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elastic buckling loads for all the buckling modes, as well as exact buckling modal shape functions.

Onah *et al* [23] used the single finite Fourier sine integral transform method for the elastic buckling analysis of uniaxially compressed Kirchhoff plate with two opposite edges simply supported and the other edges clamped (CCSS Kirchhoff plate). They found that the single finite Fourier sine transform method converts the BVP to an integral equation which further simplified to ODE. They obtained characteristic buckling equations corresponding to the exact solutions to the problem.

THEORETICAL FRAMEWORK

The study considers a rectangular Kirchhoff plate of length a, width b and thickness, t subjected to uniaxial compressive load N_x as shown in Figure-1. The elastic buckling plate theory used is expressed in the rectangular Cartesian coordinates x, y, z where x and y are coincident with the middle plane, and z points downward from the middle plane.

The Kirchhoff assumptions used are:

a) the deflections are small and maximum deflections are smaller than the plate thickness

b) the middle plane is inextensible and remains a neutral surface

c) the plate thickness is small compared to the in-plane dimensions

d) plane cross-sections of the plate rotate during flexure, but remains orthogonal to the middle surface and do not distort, and hence stresses and strains are proportional to their distance from the neutral surface

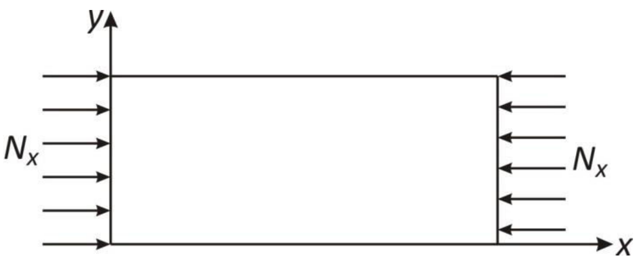

Figure-1. Kirchhoff plate subject to uniaxial compressive force N_x

The displacement field components are given by:

$$\bar{u}(x,y,z) = -z\frac{\partial w}{\partial x} \quad (1)$$

$$\bar{v}(x,y,z) = -z\frac{\partial w}{\partial y} \quad (2)$$

$$\bar{w}(x,y,z) = w(x,y) \quad (3)$$

where $\bar{u}, \bar{v}, \bar{w}$ are displacement components along the x, y, z Cartesian coordinate axis respectively. $w(x, y)$ is the transverse deflection of arbitrary point (x, y) on the middle plane $(z = 0)$.

The strain fields are obtained from the strain-displacements relations of the small displacement elasticity theory as follows:

$$\varepsilon_{xx} = \frac{\partial \bar{u}}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \quad (4)$$

$$\varepsilon_{yy} = \frac{\partial \bar{v}}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \quad (5)$$

$$\gamma_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} \quad (6)$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \quad (7)$$

$$\gamma_{xz} = \frac{\partial \bar{u}}{\partial z} + \frac{\partial w}{\partial x} = \frac{-\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0 \quad (8)$$

$$\gamma_{yz} = \frac{\partial \bar{v}}{\partial z} + \frac{\partial w}{\partial y} = \frac{-\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0 \quad (9)$$

where ε_{xx}, ε_{yy}, ε_{zz}, are normal strains, γ_{xy}, γ_{xz}, γ_{yz} are shear strains.

The virtual strain energy is given by the volume integral:

$$\partial u = \int_{R^2} \left[\int_{-t/2}^{t/2} (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \tau_{xy}\delta\gamma_{xy}) dz \right] dxdy \ldots (10)$$

$$\partial u = -\int_{R^2} \left(M_{xx}\frac{\partial^2 \delta w}{\partial x^2} + M_{yy}\frac{\partial^2 \delta w}{\partial y^2} + 2M_{xy}\frac{\partial^2 \delta w}{\partial x \partial y} \right) dxdy \quad (11)$$

where R^2 is the two dimensional domain of the middle surface; M_{xx}, M_{yy}, M_{xy} are the bending and twisting moments per unit length; and are given by:

$$M_{xx} = \int_{-t/2}^{t/2} \sigma_{xx} z\, dz \quad (12)$$

$$M_{yy} = \int_{-t/2}^{t/2} \sigma_{yy} z\, dz \quad (13)$$

$$M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z\, dz \quad (14)$$

The work done by the applied in-plane load N_x due to the displacement w is

$$w = -\frac{1}{2}\iint_{R^2} N_x \left(\frac{\partial w}{\partial x}\right)^2 dxdy \tag{15}$$

The virtual work δW due to the in-plane load N_x is thus:

$$\delta W = \iint_{R^2} N_x \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} dxdy \tag{16}$$

By the principle of virtual displacements,

$$\delta\Pi = \delta U - \delta W = 0 \tag{17}$$

$$\delta\Pi = -\iint_{R^2}\left(M_{xx}\frac{\partial^2 \delta w}{\partial x^2} + M_{yy}\frac{\partial^2 \delta w}{\partial y^2} + 2M_{xy}\frac{\partial^2 \delta w}{\partial x \partial y} + N_x \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right) dxdy = 0 \tag{18}$$

From the divergence theorem,

$$\delta\Pi = -\iint_{R^2}\left(\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} - N_x \frac{\partial^2 w}{\partial x^2}\right)\delta w dxdy -$$
$$\oint_\Gamma \left[(M_{xx}n_x + M_{xy}n_y)\frac{\partial \delta w}{\partial x} + (M_{xy}n_x + M_{yy}n_y)\frac{\partial \delta w}{\partial y}\right] ds +$$
$$\oint_\Gamma \left[\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - N_x \frac{\partial w}{\partial x}\right)n_x + \left(\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x}\right)n_y\right]\delta w ds = 0 \tag{19}$$

where n_x, n_y are the direction cosines of the unit normal n on the boundary Γ, and ds is the incremental length along the boundary. δw is arbitrary in R^2, and the domain integration is independent of the contour integration, on the boundary, the PDE obtained is for the domain R^2 is:

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} - N_x \frac{\partial^2 w}{\partial x^2} = 0 \tag{20}$$

For homogeneous isotropic plates, the Hooke's generalized stress – strain relations are:

$$\sigma_{xx} = \frac{E}{1-\mu^2}(\varepsilon_{xx} + \mu\varepsilon_{yy}) \tag{21}$$

$$\sigma_{yy} = \frac{E}{1-\mu^2}(\varepsilon_{yy} + \mu\varepsilon_{xx}) \tag{22}$$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy} \tag{23}$$

where E is the Young's modulus of elasticity, G is the shear modulus and μ is the Poisson's ratio.

Then, substitution of Equations (21-23) into Equations (12-14) gives:

$$M_{xx} = \int_{-t/2}^{t/2} \frac{E}{1-\mu^2}(\varepsilon_{xx} + \mu\varepsilon_{yy}) z\, dz = \frac{E}{1-\mu^2}\int_{-t/2}^{t/2}\left(-z\frac{\partial^2 w}{\partial x^2} - \mu z \frac{\partial^2 w}{\partial y^2}\right) z\, dz$$
$$= -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) \tag{24}$$

$$M_{yy} = \int_{-t/2}^{t/2} \frac{E}{1-\mu^2}\left(-z\frac{\partial^2 w}{\partial y^2} - \mu\frac{\partial^2 w}{\partial x^2}\right) z dz =$$
$$-D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) \tag{25}$$

$$M_{xy} = G\int_{-t/2}^{t/2} -2z\frac{\partial^2 w}{\partial x \partial y} z\, dz = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y} \tag{26}$$

$$\text{where } D = \int_{-t/2}^{t/2} \frac{Ez^2}{1-\mu^2} dz = \frac{Et^3}{12(1-\mu^2)} \tag{27}$$

The elastic buckling equation for the considered problem is obtained in terms of deflection, w by introducing the expressions for M_{xx}, M_{yy} and M_{xy} in terms of $w(x, y)$ in Equation (20):

$$\frac{\partial^2}{\partial x^2}\left\{-D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)\right\} + \frac{\partial^2}{\partial y^2}\left\{-D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right)\right\} + 2\frac{\partial^2}{\partial x \partial y}\left\{-D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}\right\} - N_x\frac{\partial^2 w}{\partial x^2} = 0 \tag{28}$$

Hence,

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + N_x\frac{\partial^2 w}{\partial x^2} = 0 \tag{29}$$

$$\text{or, } \nabla^4 w(x, y) + \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2}(x, y) = 0 \tag{30}$$

$$\text{where } \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{31}$$

∇^4 is the biharmonic partial differential operator.

Clamped Edge Boundary Conditions

At a clamped edge C, the deflection and slope are required to vanish. Along a clamped edge, $x = a$, (say)

$$w(x = a, y) = 0 \tag{32}$$

$$\frac{\partial w}{\partial x}(x = a, y) = 0 \tag{33}$$

Simply supported edges, S

Along a simply supported edge S, $x = a$ (say) the deflection and bending moment are both zero.

$$w(x = a, y) = 0 \tag{34}$$

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$$M_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{x=a,y} = 0 \qquad (35)$$

Free edge *F*

The bending moment and vertical shear force along a free edge will both vanish.

METHODOLOGY

The considered Kirchhoff plate buckling problem is simply supported at $x = 0$, $x = a$, $y = 0$ and free at $y = b$ with the origin of coordinates taken at one corner of the plate. The uniform compression force is applied at $x = 0$, and $x = a$. The deflection function is chosen by the Vlasov procedure to be the linear combinations of the product of the eigenfunctions of a freely vibrating Euler - Bernoulli beam in the x - direction, and an unknown function of y denoted by $A(y)$ in the y coordinate direction. Thus,

$$w(x, y) = \sum_{n=1}^{\infty} A_n(y) \sin \frac{n\pi x}{a} \qquad (36)$$

where n is a positive integer and represents the buckling mode number.

The Galerkin-Vlasov variational equation for the considered problem becomes the following integral equation:

$$\int_0^b \int_0^a \left\{ \nabla^4 \left(\sum_{n=1}^{\infty} A_n(y) \sin \frac{n\pi x}{a} \right) + \frac{N_x}{D} \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} A_n(y) \sin \frac{n\pi x}{a} \right\} \sin \frac{m\pi x}{a}\, dxdy = 0 \qquad (37)$$

Hence,

$$\sum_{n=1}^{\infty} \int_0^b \int_0^a \left\{ \nabla^4 \left(A_n(y) \sin \frac{n\pi x}{a} \right) + \frac{N_x}{D} \frac{\partial^2}{\partial x^2} A_n(y) \sin \frac{n\pi x}{a} \right\} \sin \frac{m\pi x}{a}\, dxdy = 0 \qquad (38)$$

$$\sum_{n=1}^{\infty} \int_0^b \int_0^a \left\{ A_n^{iv}(y) - 2\left(\frac{n\pi}{a}\right)^2 A_n''(y) + \left(\left(\frac{n\pi}{a}\right)^4 - \frac{N_x}{D}\left(\frac{n\pi}{a}\right)^2 \right) A_n(y) \right\} \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a}\, dxdy = 0 \qquad (39)$$

$$\sum_{n=1}^{\infty} \int_0^b \left(A_n^{iv}(y) - 2\left(\frac{n\pi}{a}\right)^2 A_n''(y) + \left(\left(\frac{n\pi}{a}\right)^4 - \frac{N_x}{D}\left(\frac{n\pi}{a}\right)^2 \right) A_n(y) \right) I_{mn}\, dy = 0 \qquad (40)$$

$$\text{where } I_{mn} = \int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a}\, dx \qquad (41)$$

By the orthogonality properties of the sine functions, which is the basis function in the x coordinate direction,

$$I_{mn} = 0, \quad \text{if } m \neq n \qquad (42)$$

$$I_{mn} = \frac{a}{2} \quad \text{if } m = n \qquad (43)$$

RESULTS

The unknown function $A_n(y)$ is found by solving the system of ordinary differential equations which results from Equation(40). Hence at the nth buckling mode,

$$A_n^{iv}(y) - 2\left(\frac{n\pi}{a}\right)^2 A_n''(y) + \left(\left(\frac{m\pi}{a}\right)^4 - \frac{N_x}{D}\left(\frac{n\pi}{a}\right)^2 \right) A_n(y) = 0 \qquad (44)$$

$$\text{Let } \frac{n\pi}{a} = \beta_n \qquad (45)$$

The fourth order ordinary differential equation (ODE) can be expressed in compact form as:

$$A_n^{iv}(y) - 2\beta_n^2 A_n''(y) + \left(\beta_n^4 - \frac{N_x}{D}\beta_n^2 \right) A_n(y) = 0 \qquad (46)$$

The solution, using the method of differential operators or the method of undetermined parameters is found as:

$$A_n(y) = c_1 \cosh \alpha_1 y + c_2 \sinh \alpha_1 y + c_3 \cos \alpha_2 y + c_4 \sin \alpha_2 y \qquad (47)$$

Where

$$\alpha_1 = \pm\left(\sqrt{\beta_n^2 \frac{N_x}{D}} + \beta_n^2 \right)^{1/2} \qquad (48)$$

$$\alpha_1 = \pm\left(\sqrt{\beta_n^2 \frac{N_x}{D}} - \beta_n^2 \right)^{1/2} \qquad (49)$$

Since, $N_x > D\left(\frac{n\pi}{a}\right)^2$ due to the geometric restraints or constraints along $y = 0$, and $y = b$. Constants c_1, c_2, c_3 and c_4 are the four constants of integration which are determined by enforcement of the boundary conditions along $y = 0$, and $y = b$ edges. The problem considered a thin SSSF plate restrained and loaded as shown in Figure-2.

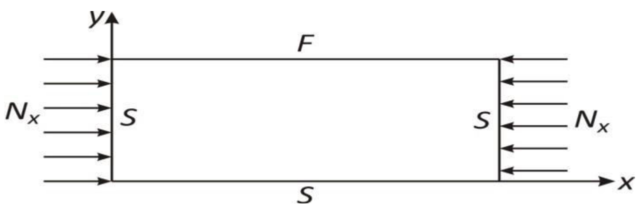

Figure-2. SSSF Kirchhoff plate under uniaxial compressive load N_x.

The boundary conditions are:

$$w(x, y = 0) = 0 \qquad (50)$$

$$\therefore A_n(y = 0) = 0 \qquad (51)$$

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$$M_{yy}(x, y=0)=0 \tag{52}$$

$$\therefore A_n''(y=0)-\mu\left(\frac{m\pi}{a}\right)^2 A_n(y=0)=$$

$$A_n''(y=0)-\mu\beta_n^2 A_n(y=0)=0 \tag{53}$$

$$M_{yy}(y=b)=0 \tag{54}$$

$$A_n''(y=b)-\mu\left(\frac{n\pi}{a}\right)^2 A_n(y=b)=$$

$$A_n'''(y=b)-\mu\beta_n^2 A_n(y=b)=0 \tag{55}$$

At the free edge, $y = b$,

$$V_y=-D\left(\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right)\Bigg|_{(x,y=b)}=0 \tag{56}$$

$$A_n'''(y=b)-(2-\mu)\left(\frac{n\pi}{a}\right)^2 A_n'(y=b)=$$

$$A_n'''(y=b)-(2-\mu)\beta_n^2 A_n'(y=b)=0 \tag{57}$$

Enforcement of Boundary Conditions

Application of the boundary conditions gives:

$$A_n(y=0)=c_1+c_3=0 \tag{58}$$

$$A_n''(y=0)-\mu\beta_n^2 A_n(y=0)=c_1\alpha_1^2-c_3\alpha_2^2-\mu\beta_n^2(c_1+c_3)=0 \tag{59}$$

$$\therefore c_1(\alpha_1^2-\mu\beta_n^2)-c_3(\alpha_2^2+\mu\beta_n^2)=0 \tag{60}$$

$$A_n''(y=b)-\mu\beta_n^2 A_n(y=b)=c_1(\alpha_1^2-\mu\beta_n^2)\cosh\alpha_1 b+c_2(\alpha_1^2-\mu\beta_n^2)\sinh\alpha_1 b-$$

$$c_3(\alpha_2^2+\mu\beta_n^2)\cos\alpha_2 b-c_4(\alpha_2^2+\mu\beta_n^2)\sin\alpha_2 b=0\ \ldots \tag{61}$$

$$A_n'''(y=b)-(2-\mu)\beta_n^2 A_n'(y=b)=c_1(\alpha_1^3\sinh\alpha_1 b-(2-\mu)\beta_n^2\alpha_1\sinh\alpha_1 b)+$$
$$c_2(\alpha_1^3\cosh\alpha_1 b-(2-\mu)\beta_n^2\alpha_1\cosh\alpha_1 b)+c_3(\alpha_2^3\sin\alpha_2 b+(2-\mu)\beta_n^2\alpha_2\sin\alpha_2 b)-$$
$$c_4(\alpha_2^3\cos\alpha_2 b+(2-\mu)\beta_n^2\alpha_2\cos\alpha_2 b)=0 \tag{62}$$

Simplifying,

$$c_1\alpha_1(\alpha_1^2-(2-\mu)\beta_n^2)\sinh\alpha_1 b+c_2\alpha_1(\alpha_1^2-(2-\mu)\beta_n^2)\cosh\alpha_1 b+$$
$$c_3\alpha_2(\alpha_2^2+(2-\mu)\beta_n^2)\sin\alpha_2 b-c_4\alpha_2(\alpha_2^2+(2-\mu)\beta_n^2)\cos\alpha_2 b=0 \tag{63}$$

The homogeneous algebraic equations obtained by the enforcement of the four sets of boundary conditions are given by Equations (58), (60), (61) and (63). From Equation (58)

$$c_1=-c_3 \tag{64}$$

Substitution of Equation (64) into Equation (60) gives:

$$c_1(\alpha_1^2-\mu\beta_n^2)+c_1(\alpha_2^2+\mu\beta_n^2)=0 \tag{65}$$

$$c_1(\alpha_1^2+\alpha_2^2)=0 \tag{66}$$

$$(\alpha_1^2+\alpha_2^2)\neq 0$$

$$\therefore c_1=0 \tag{67}$$

$$c_3=0 \tag{68}$$

Then the system of homogeneous algebraic equations reduces to:

$$c_2(\alpha_1^2-\mu\beta_n^2)\sinh\alpha_1 b-c_4(\alpha_2^2+\mu\beta_n^2)\sin\alpha_2 b=0\ \ldots \tag{69}$$

$$c_2\alpha_1(\alpha_1^2-(2-\mu)\beta_n^2)\cosh\alpha_1 b-c_4\alpha_2(\alpha_2^2+(2-\mu)\beta_n^2)\cos\alpha_2 b=0 \tag{70}$$

Let $\lambda_1=\alpha_1^2-\mu\beta_n^2$ (71)

$$\lambda_2=\alpha_2^2+\mu\beta_n^2 \tag{72}$$

It can be shown that:

$$\alpha_1^2-\mu\beta_n^2=\alpha_2^2+(2-\mu)\beta_n^2=\lambda_1 \tag{73}$$

$$\alpha_2^2+\mu\beta_n^2=\alpha_1^2-(2-\mu)\beta_n^2=\lambda_2 \tag{74}$$

Then we obtain:

$$c_2\lambda_1\sinh\alpha_1 b-c_4\lambda_2\sin\alpha_2 b=0 \tag{75}$$

and $c_2\alpha_1\lambda_2\cosh\alpha_1 b-c_4\alpha_2\lambda_1\cos\alpha_2 b=0$ (76)

In matrix form,

$$\begin{pmatrix}\lambda_1\sinh\alpha_1 b & -\lambda_2\sin\alpha_2 b\\ \alpha_1\lambda_2\cosh\alpha_1 b & -\alpha_2\lambda_1\cos\alpha_2 b\end{pmatrix}\begin{pmatrix}c_2\\ c_4\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{77}$$

For nontrivial solutions, $\begin{pmatrix}c_2\\ c_4\end{pmatrix}\neq 0$

The condition for nontrivial solutions gives the characteristic buckling equation as:

$$\begin{vmatrix}\lambda_1\sinh\alpha_1 b & -\lambda_2\sin\alpha_2 b\\ \alpha_1\lambda_2\cosh\alpha_1 b & -\alpha_2\lambda_1\cos\alpha_2 b\end{vmatrix}=0 \tag{78}$$

Expansion of the determinant gives the characteristic buckling equation as:

$$-\alpha_2\lambda_1^2\sinh\alpha_1 b\cos\alpha_2 b+\alpha_1\lambda_2^2\cosh\alpha_1 b\sin\alpha_2 b=0 \tag{79}$$

The characteristic buckling equation is expressed explicitly as:

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$$-\alpha_2(\alpha_1^2-\mu\beta_n^2)^2\sinh\alpha_1 b\cos\alpha_2 b+\alpha_1(\alpha_2^2+\mu\beta_n^2)^2\cosh\alpha_1 b\sin\alpha_2 b=0 \quad (80)$$

The characteristic buckling equation is solved using computer based iterative methods to generate solutions for α_1 and α_2 from which, using Equations (48) and(49), the elastic buckling loads N_x are evaluated for $\mu = 0.25$, and for buckling modes n = 1, 2 and various values of the plate aspect ratio (a/b).

The general expression for the buckling mode shape is found as:

$$w(x,y)=(\lambda_1\sinh\alpha_1 b\sin\alpha_2 y+\lambda_2\sinh\alpha_1 y\sin\alpha_2 b)\sin\beta_n x \quad (81)$$

where, $0\le x\le a$, $0\le y\le b$.

The critical elastic buckling load coefficients obtained by Timoshenko [24] for the considered elastic thin plate buckling problem, and the buckling load coefficients obtained for the SSSF Kirchhoff plate under uniaxial compressive load N_x at the two opposite simply supported edges x = 0, x = a are presented in Table-1 for buckling modes, n = 1, and n = 2.

Table-1. Non dimensional buckling loads or buckling load coefficients for SSSF Kirchhoff plate subjected to uniaxial compressive force N_x along the edges x = 0, x = a.

a/b	***K*(*a/b*) for *n* =1 Present results**	***K*(*a/b*) for *n* =2 Present results**	**Timoshenko [24] *K*(*a/b*) for *n* =1**	**% Difference of present results (*n* = 1) and Timoshenko [24]**
0.4	6.6367	25.2899		
0.6	3.1921	11.4675		
0.8	1.9894	6.3667		
1.0	1.4342	4.4036	1.44	–0.403
1.5	0.8880	2.2022		
2	0.6979	1.4342	0.698	–0.0205
2.5	0.6104	1.0798	0.610	0.0656
3	0.5630	0.8879	0.564	–0.1773
3.5	0.5345	0.7726		
4	0.5161	0.6979	0.517	–0.1741
4.5	0.5034	0.6469		
5	0.4944	0.6104		
5.5	0.4877	0.5835		
6	0.4826	0.5630		

In Table-1,

$$K\left(\frac{a}{b}\right)=\frac{N_{xx}b^2}{\pi^2 D} \quad (82)$$

and, $$N_{xx}=K\left(\frac{a}{b}\right)\left(\frac{\pi^2 D}{b^2}\right) \quad (83)$$

The critical elastic buckling load is found for the first buckling mode n =1, since $K(a/b)$ for n = 1 is less than $K(a/b)$ for n = 2 for all considered values of a/b. The buckling mode function at the critical buckling mode, for which n = 1 is obtained is:

$$w(x,y)=(\lambda_1\sinh\alpha_1 b\sin\alpha_2 y+\lambda_2\sinh\alpha_1 y\sin\alpha_2 b)\sin\frac{\pi x}{a} \quad (84)$$

DISCUSSIONS

The elastic buckling problem of a rectangular SSSF Kirchhoff plate subjected to uniaxial compressive load N_x on the two opposite simply supported edges x = 0 and x = a has been successfully solved in this work using the Galerkin-Vlasov variational method. The problem is a boundary value problem (BVP) of elastic stability that could be derived using equilibrium or variational approaches. As a BVP, it is represented by the partial differential equation - Equation (30) - and boundary conditions, determined by the edge support conditions. For the SSSF plate considered, the Vlasov procedure allowed the use of the eigenfunctions of a vibrating simply supported Euler - Bernoulli beam as the basis functions in the x coordinate direction. The deflection function $w(x, y)$ used has been expressed in Equation (36) as a linear combination of the products of unknown function $A_n(y)$ and the eigenfunctions in the x direction. Thus, the boundary conditions along the x = 0 and x = a axes are fully satisfied by $w(x, y)$.

The Galerkin-Vlasov variational equation is expressed as Equation (37) and upon simplification the integral equation reduced to a system of n fourth order ordinary differential equations. Equation (44). The solution of Equation (44) has given the unknown function $A_n(y)$ as Equation (47) which is in terms of four unknown integration constants. The boundary conditions of the SSSF plate has been used to obtain a system of four homogeneous equations - Equations (58), (60), (61) and (63). Solution of the resulting equations has reduced to a system of two equations given in matrix form as Equation (77). The characteristic buckling equation obtained using the condition for nontrivial solution is found as Equation (80).

The characteristic buckling equation is solved using computer based iterative methods to obtain the elastic buckling loads for the first two modes $n = 1$ and $n = 2$ and for $\mu = 0.25$, for various values of the plate aspect ratio (a/b). The buckling load coefficients obtained for the first two buckling modes for various values of a/b are presented in Table-1.

Table-1 reveals that critical elastic buckling occurs at the first buckling mode for which $n = 1$ since the buckling load coefficients for $n= 1$ are smaller than those for the corresponding plate aspect ratio (a/b) for $n = 2$. The buckling mode shape function has been obtained as Equation (81) and as Equation (84) for the critical buckling mode. The solutions obtained are comparable with solutions obtained by Timoshenko [24] who used various other methods. The difference between the critical elastic buckling load obtained in this study and the Timoshenko's results vary from - 0.403% for $a/b = 1$ to - 0.1741% for $a/b = 4$; and are generally less than 0.5%.

CONCLUSIONS

In conclusion,

a) The Galerkin-Vlasov method converts the BVP of elastic buckling of Kirchhoff plate to an integral equation.

b) The use of the eigenfunctions of a vibrating simply supported Euler-Bernoulli beam in the x coordinate direction simplifies the integral equation problem due to the orthogonality of the eigenfunctions.

c) The integral equation is reduced upon simplification to a homogeneous fourth order ODE in the unknown function $A_n(y)$.

d) Enforcement of the boundary conditions along the $y = 0$ and $y = b$ edges is found to result in a system of four homogeneous equations in terms of the unknown integration constants.

e) The conditions for nontrivial solution is found to give the characteristic elastic buckling equation as a transcendental equation which is solved for α_1, α_2 and hence for N_x using iterative methods, and computational software tools.

f) Critical elastic buckling load corresponds to the first buckling mode.

g) The critical elastic buckling load coefficients obtained agrees with previous results obtained by Timoshenko, and the differences are less than 0.5%.

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