NUMERICAL SOLUTION OF POINT KINETIC EQUATIONS USING RK2-2ST WITH ADIABATIC DOPPLER EFFECTS CONSIDERING COMPENSATED RAMP REACTIVITY

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ABSTRACT
The second order second stage stochastic Runge-Kutta method (RK2-2st) is implemented for solving stochastic point kinetic equations with Newtonian temperature feedback effects, taking into consideration external ramp reactivity. The feedback temperature is included in the reactivity; it is an entry variable of point kinetic equations. Doppler feedback in thermonic reactors is mainly due to epithermal capture resonances in non-fissionable combustible isotopes. Different numerical experiments have been carried out in which calculations are made of reactivity, mean values and standard deviation of neutron density, and the concentration of delayed neutron precursors. The results obtained are compared with other methods reported in literature, and it is found that the method proposed is sufficiently precise to give a solution to the stochastic point kinetic equations of an adiabatic reactor.

Keywords: neutron density, concentration of delayed neutron precursor density, doppler effects, reactivity, runge-kutta stochastic method.

INTRODUCTION
Stochastic point kinetic equations [1-3] are a rigid system of ordinary differential equations which describe neutron population density and the concentration of delayed neutron precursors. This rigidity represents a great challenge in solving the system of equations and obtaining efficient and accurate results. In a nuclear reactor the fission process is carried out which occurs due to the high probability of interaction between the thermal neutrons with the fissionable material [4]; this nuclear reactor is closely coupled according to time.

Stochastic point kinetic equations present the form of stochastic differential equations (SDE) [5], which describe the dynamic evolution which implies randomness in neutron density functions and the concentration of delayed neutron precursors. Not only do SDEs have application in the field of nuclear physics, it has also been very useful in studying different phenomena that arise in various fields of application, such as in population genetics, finance, communication systems among others.

There is a model that describes the dynamic of nuclear reactors in which the effects of temperature feedback are especially considered [6-7]. Based on these models it is possible to estimate the transient behavior of neutron density and the concentration of delayed neutron precursors, enabling timely control of the nuclear reactor. Reactivity is affected by temperature feedback, and this is one of the most important variables because it constitutes a nonlinear term with the product of neutron density. Reactivity is an input variable of point kinetic equations, and for this work it is modeled as a reactivity ramp.

It has become a challenge to give a solution to the deterministic equations of the point nuclear reactor, for which there have been many attempts at detailed studies. Some of these are: the Power series solutions (PWS) method [8] which is based on the partial approximation of reactivity and functions of sources neutrons, converged accelerated Taylor series (CATS) method[9] based on accelerators of non-linear and linear convergence, and which is the best method of estimation for solving point kinetic equations, enhanced piecewise constant approximation (EPCA) method [10], which is a semi-analytical method which solves point kinetic equations with a technique which repeatedly corrects the error in the source term, the ITS2 method [11] is an explicit method of solution based on Taylor’s lower order series expansions, the generalized Adams-Bashforth-Moulton Method [12], the 8th-order Adams-Bashforth-Moulton (ABM8) method temperature [13] which are predictor-corrector methods with their respective modifiers which increase precision, and finally the Euler Maruyama method, the derivative-free Milstein method [6] and Taylors strong-order 1.5 method [14] which are methods which give solutions to point kinetic stochastic equations with temperature feedback effects.

In this work, an analysis of the efficiency and accuracy of the stochastic second order two-stage Runge-Kutta method (RK2-2st) is performed to provide solutions to stochastic point kinetic equations with temperature feedback effects, considering external ramp reactivity. Several numerical experiments are proposed which are compared with different methods reported in literature, in which the CATS method is one of the reference methods for its high degree of accuracy.

THEORETICAL CONSIDERATIONS STOCHASTIC POINT KINETIC EQUATIONS
The deterministic point kinetic equations with effects of temperature feedback with m group of delayed neutron precursors are given by the following system
\[ \frac{dn(t)}{dt} = \frac{\rho(t) - \beta}{\Lambda} n(t) + \sum_{i=1}^{m} \lambda_i c_i(t) + Q(t) \]  

(1)

\[ \frac{dc_i(t)}{dt} = \frac{\beta_i}{\Lambda} n(t) + \lambda_i c_i(t) \]  

(2)

\[ \rho(t) = a(t-t_0) - b \int_{t_0}^{t} n(t')dt' \]  

(3)

\[ \frac{d\rho(t)}{dt} = a - b n(t) \]  

(4)

Subject to the initial conditions

\[ n(t_0) = n_0 \]  

(5)

\[ c_i(t_0) = c_{i0} \]  

(6)

\[ \rho(t_0) = \rho_0 \]  

(7)

Where \( n_0 \) is the neutron density, \( c_{i0} \) is the delayed neutron precursor density of the \( i \)-th group of precursors, \( \beta \) is the fraction of delayed neutrons of the \( i \)-th group of precursors, \( \beta \) is the total fraction of delayed neutrons, \( \lambda_i \) is the decay constant of the \( i \)-th group of precursors, \( v \) is the fission velocity of the neutrons, \( \Lambda \) is time of neutron generation, \( Q(t) \) is the external neutron source, \( \rho(t) \) is the reactivity, \( a \) is the impressed reactivity variation, \( b \) is the shutdown coefficient.

Equations (1-3) are a system of rigidly coupled non-linear differential equations in which function \( \rho(t) \) has a dependency on neutron density \( n_0 \). These parameters represent the effects of Newtonian temperature feedback with ramp reactivity insertions. Nevertheless, in nature, the dynamic processes of a nuclear reactor are stochastic. [2-3]:

\[ d\hat{x}(t) = \left[ A\hat{x}(t) + \hat{Q}(t) \right] dt + \mathcal{W}(t) \]  

(8)

\[ \hat{x}(t_0) = \begin{bmatrix} n(t_0) \\ c_{i(t_0)} \\ \vdots \\ c_{n(t_0)} \end{bmatrix}, \quad \hat{Q}(t_0) = \begin{bmatrix} q(t_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathcal{W}(t_0) = \begin{bmatrix} W_{1(t_0)} \\ W_{2(t_0)} \\ \vdots \\ W_{n(t_0)} \end{bmatrix} \]

\( \mathcal{W}(t) \) are Wiener processes, \( A \) is the coefficient matrix which is given in the following way

\[
A = \begin{bmatrix}
\rho_0 - \beta & \lambda_1 & \lambda_2 & \cdots & \lambda_m \\
\frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & \cdots & 0 \\
\frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\beta_m}{\Lambda} & 0 & 0 & \cdots & -\lambda_m \\
\end{bmatrix}
\]

(9)

\( B \) is the covariance matrix which presents an approximation of pulsed neutrons [15], defined as

\[
B = \begin{bmatrix}
\zeta & a_1 & a_2 & \cdots & a_r \\
a_1 & b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\
a_2 & b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_r & b_{r,1} & b_{r,2} & \cdots & b_{r,m} \\
\end{bmatrix}
\]

(10)

where,

\[
\zeta = \left[ \frac{\rho(t)-\beta}{\Lambda} \right] \left[ 1 - \rho(t) + 2\beta + (1 - \beta) v \right] n(t) + \sum_{i=1}^{m} \lambda_i c_i(t) + q(t)
\]

\[
a_i = \beta \left[ (1 - \beta) v - \Lambda \right] n(t) - \lambda_i c_i(t)
\]

\[
b_{i,j} = \beta \left[ \frac{\rho(t)-\beta}{\Lambda} \right] n(t) + \delta_{i,j} \lambda_i c_i(t)
\]

\[
\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j 
\end{cases}
\]

By providing solutions to the stochastic point kinetic equations given by equation (8) the temporal dependence of the processes involved in the temporal behavior of neutron flow in the reactor, such as neutron density and concentration of delayed neutron precursors can be found. When in equation (8) the covariance matrix \( B \) is equal to zero, replicating the system of equations of the deterministic case of the point kinetic equations given by equation (1) and equation (2).

SECOND ORDER, TWO STAGES STOCHASTIC RUNGE-KUTTA SCHEME

The stochastic point kinetic equations given by equation (8) correspond to the Itô process in \( t \in [t_0,T] \) which presents a stochastic differential equation structure (SDE) given as follows:

\[ dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t \]  

(11)

With the initial condition \( X_0 = x \). Where \( X \) is a stochastic process which has continuous sample paths.
which take values in $R^d$, for a number of $d$ positive integer, $b$ and $\sigma$ are functions with values of given coefficients and $W$ is a Brownian movement, which must comply with the following conditions:

a) $W(0) = 0$ with probability 1.

b) For $0 \leq s < t \leq T$ the random variable given by the increase $W(t) - W(s)$ for a normal distribution with zero mean and variance $t - s$.

$$\bar{X}_{n+1} = \bar{X}_n + a(t_n, \bar{X}_n) + \frac{1}{2} a(b(t_n + \Delta, \eta) \Delta + \frac{1}{2} b(t_n + \Delta, \eta) \Delta W_n - \frac{1}{2} b \frac{\partial b}{\partial x} \Delta$$

where

$$a = a(t_n, \bar{X}_n)$$

$$b = b(t_n, \bar{X}_n)$$

$$\bar{X}_{n+1} = \bar{X}_n + a(t_n, \bar{X}_n) + \frac{1}{2} b(t_n + \Delta, \eta) \Delta W_n + \frac{1}{2} b(t_n, \bar{X}_n) \Delta \bar{W}_n$$

$$\eta = \bar{X}_n + a(t_n, \bar{X}_n) \Delta + b(t_n, \bar{X}_n) \Delta W_n$$

Comparing Taylor’s second order scheme of equation (13) with Runge-Kutta second order, second stage method given by equation (12) it can be seen that the RK2-2st is an easily applied numerical method since it requires he calculation of only one derivative.

In this study a solution will be given to the stochastic equations which model the dynamic of a nuclear reactor with feedback effects of Newtonian temperature described by equation (8) and equation (3) with the RK2-2st method given by equation (12).

RESULTS

This section will discuss the efficiency and accuracy of the proposed RK2-2st method for understanding neutron density, concentration of delayed neutron precursors and reactivity when temperature feedback effects exist for the case of insertions of ramp reactivity. Different numerical experiments will be carried out which present the same initial conditions (in $t_0 = 0$) for neutron density and the concentration of delayed neutron precursors, $n_0 = 1 \text{ n/cm}^3$ and $C_i(0) = \frac{\beta \rho_i(0)}{\lambda_i \Lambda}$.

Parameters of the groups of delayed neutron precursors of the nuclear reactor of graphite $U^{235}$, which correspond to the decay constant $a$: $\lambda_i = [0.0124, 0.0305, 0.111, 0.301, 1.13, 3.0] s^{-1}$, fraction of delayed neutrons $\beta_i = [0.00021, 0.00141, 0.00127, 0.00255, 0.00074, 0.00027]$, total fraction of delayed neutrons $\beta = 0.00645$, time of neutron generation $\Lambda = 5.0 \times 10^{-5} s$. In the programmed numerical simulations the command ‘state’ was used to generate the pseudorandom numbers with normal distribution and seed 1000, and a fixed step size $h = 10^{-3}$, using 500 samples for each experiment.

Compensated Ramp Change

For a compensated response with ramp reactivity inserts described by equation (4) the impressed reactivity variation $a = 0.003 s^{-1}$ is considered, using a fixed shutdown coefficient $b = 10^{-11} \text{ cm}^3 / s$. Table-1 shows the mean values ($E$) and standard deviation ($\sigma$) for neutron density, the concentration of the sum of delayed neutron precursors and reactivity. The mean values of neutron density and reactivity are compared ITS2 [21]. Figure-1a shows the variation of neutron density according to time. Figure-1b shows the variation of the sum of the density of delayed neutron precursors according to time, and Figure-1c shows the behaviour of reactivity through time. For the graphics two random samples were taken of the numerical experiments (purple and blue line) and the average of the 500 samples (red line). It can be seen that the RK2-2st method presents values which are very close to those reported with method ITS2.
The peak of neutron density and the respective time in which the peak is generated are recorded in Table-2. The numerical results obtained with RK2-2st are compared with the deterministic methods the power series methods [19], CATS [20], ITS2 [21] and ABM8 [22]. The results of neutron density and reactivity confirm the extreme precision of the CATS method. Comparing method RK2-2st it is shown that this is accurate for studying compensated response for ramp reactivity insertions.

Table-1. Mean values and standard deviation of neutron density and the sum of precursor density, and reactivity using compensated response for ramp reactivity with \( a = 0.003s^{-1} \) and \( b = 10^{-11} cm^2 / s \).

<table>
<thead>
<tr>
<th>Table</th>
<th>ITS2</th>
<th>RK2-2st</th>
</tr>
</thead>
<tbody>
<tr>
<td>t(s)</td>
<td>( E[n_{(t)}] )</td>
<td>( \rho_{(t)} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>1.3247E+00</td>
<td>2.3256,E+08</td>
</tr>
<tr>
<td>1</td>
<td>2.0532E+00</td>
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</tr>
<tr>
<td>1.5</td>
<td>4.3472E+00</td>
<td>6.9767,E+08</td>
</tr>
<tr>
<td>2</td>
<td>2.3921E+01</td>
<td>9.3023,E+08</td>
</tr>
<tr>
<td>2.5</td>
<td>1.4390E+04</td>
<td>1.1628,E+07</td>
</tr>
<tr>
<td>2.7</td>
<td>4.6419E+06</td>
<td>1.2556,E+07</td>
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<tr>
<td>2.8</td>
<td>1.9053E+08</td>
<td>1.2946,E+07</td>
</tr>
<tr>
<td>2.9</td>
<td>4.8596E+09</td>
<td>1.0595,E+07</td>
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<tr>
<td>3</td>
<td>8.0571E+08</td>
<td>6.4673,E+08</td>
</tr>
<tr>
<td>4</td>
<td>3.4133E+08</td>
<td>5.0440,E+08</td>
</tr>
<tr>
<td>5</td>
<td>3.2933E+08</td>
<td>4.5283,E+08</td>
</tr>
<tr>
<td>10</td>
<td>3.1459E+08</td>
<td>2.7934,E+08</td>
</tr>
</tbody>
</table>
1a) Neutron density

1b) Sum of precursor density

1c) Reactivity

Figure 1. Variation of neutron density sum of precursor density and reactivity according to time with 

\[ a = 0.003 \text{s}^{-1} \text{ and } b = 10^{-11} \text{ cm}^3 / \text{s} . \]

Table 2. Peak of neutron density and time to peak in compensated ramp with 

\[ a = 0.003 \text{s}^{-1} \text{ and } b = 10^{-11} \text{ cm}^3 / \text{s} . \]

<table>
<thead>
<tr>
<th>Method</th>
<th>Peak neutron</th>
<th>Time to peak</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power series</td>
<td>5.0537E+09</td>
<td>2.84999</td>
</tr>
<tr>
<td>CATS</td>
<td>5.1141566E+09</td>
<td>2.910581</td>
</tr>
<tr>
<td>ITS2</td>
<td>5.11415988E+09</td>
<td>2.910582</td>
</tr>
<tr>
<td>ABM8 (h=0.001s)</td>
<td>5.11375187E+09</td>
<td>2.911</td>
</tr>
<tr>
<td>RK2-2st</td>
<td>5.2394E+09</td>
<td>2.912</td>
</tr>
</tbody>
</table>

With the proposed RK2-2st method, numerical calculations are made of neutron density, the sum of delayed neutron precursor concentration and reactivity using changes in reactivity with compensated ramp, which are presented in Table 3. Reactivity values are expressed in $ (dollar).
Table-3. Mean values and standard deviation of neutron density and the sum of precursor density and reactivity for $a = 0.01\text{s}^{-1}$ and $a = 0.003\text{s}^{-1}$ with $b = 10^{-13}\text{cm}^3/\text{s}$.

<table>
<thead>
<tr>
<th>t(s)</th>
<th>$E[n_{(t)}]$</th>
<th>$\sigma[n_{(t)}]$</th>
<th>$E[c_{(t)}]$</th>
<th>$\sigma[c_{(t)}]$</th>
<th>$\rho_{(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a) $a = 0.01\text{s}^{-1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>1.0004E+00</td>
<td>3.4831E+00</td>
<td>1.6767E+03</td>
<td>1.6599E+01</td>
<td>1.5504E-01</td>
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<tr>
<td>0.5</td>
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<td>7.9123E+05</td>
<td>3.8545E+13</td>
<td>6.3865E+08</td>
<td>4.3645E-01</td>
</tr>
<tr>
<td>7.5</td>
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<td>1.3594E+05</td>
<td>5.1182E+13</td>
<td>3.0308E+08</td>
<td>3.3638E-01</td>
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<tr>
<td>10</td>
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<td>9.3704E+04</td>
<td>6.1137E+13</td>
<td>1.8360E+08</td>
<td>2.7590E-01</td>
</tr>
<tr>
<td>b) $a = 0.003\text{s}^{-1}$</td>
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</tr>
<tr>
<td>0.1</td>
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<td>3.7150E+00</td>
<td>1.6760E+03</td>
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<td>4.6512E-02</td>
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<td>3.9518E+01</td>
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<td>5.4386E+09</td>
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<td>7.1755E+06</td>
<td>1.5131E+13</td>
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</tr>
<tr>
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<td>3.1464E+10</td>
<td>1.6650E+05</td>
<td>1.8345E+13</td>
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<td>2.8053E-01</td>
</tr>
</tbody>
</table>

2a) Neutron density  
2b) Sum of precursor density  
2c) Reactivity

Figure-2. Variation of neutron density, sum of precursor density and reactivity according to time with $a = 0.001\text{s}^{-1}$ and $b = 10^{-13}\text{cm}^3/\text{s}$.
The density of neutrons obtained with the RK2-2st method using changes in compensated ramp reactivity are compared with the deterministic methods CATS [9], EPCA [10], the ITS2 method [11] and ABM method [13], reported in Table-4. It was confirmed that the RK2-2st is an effective method and sufficiently precise for insertions of ramp reactivity with temperature feedback.

**Table-4.** Neutron density for different values of \( a \) with \( b = 10^{-13} \text{ cm}^3 / \text{s} \).

<table>
<thead>
<tr>
<th>t(s)</th>
<th>EPCA</th>
<th>CATS</th>
<th>ITS2</th>
<th>ABM8 (h=0.001)</th>
<th>RK2-2st</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( a = 0.01 \text{s}^{-1} )</td>
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<tr>
<td>0.1</td>
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<td>1.1672108379E+00</td>
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</tr>
<tr>
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<td>1.0124348832E+11</td>
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<td>1.0124E+11</td>
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<tr>
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<td>( a = 0.003 \text{s}^{-1} )</td>
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</tr>
</tbody>
</table>
CONCLUSIONS

Based on numerical experiments carried out to study stochastic point kinetic equations with a nuclear reactor with adiabatic Doppler effects considering external ramp reactivity, it was found that the proposed stochastic second order second stage Runge-Kutta (RK2-2st) method is the numerical derivation method that provides a solution to the system of equations and generates values in neutron density and reactivity which are very similar to those reported by one of the most exact methods (deterministic) such as CATS. The RK2-2st method is equivalent to the second order Taylor scheme, and it is easy to implement since its scheme implies only one derivative of covariant matrix regarding to random variables. RK2-2st is an alternative and efficient method to study the adiabatic nuclear reactor.

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