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MATHEMATICAL MODEL FOR VALUING OPTIONS

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ABSTRACT

In this article, the classical theory leading to the Black-Scholes equation, widely used in the market model for valuing financial options developed. In this sense, the mathematical development from a stochastic differential equation leads to the equation, Black Scholes; therefore, the model's solution is presented. Initially, the model is transformed into the heat equation, and then it is combined with the inverse Fourier transformation, supported in the performance of the separation of variables method and the description of the solution according to the nature of the root of the characteristic polynomial. A solution of the *call* option is achieved. From this, through the parity relation, the value of a *put* option is also found. Finally, the volatility parameter associated with the model is estimated through the classic and Bayesian approach, where it was shown that when evaluating the premium or price of the option from the volatility estimate from the Bayesian approach, it presents a lower proportion of risk to what happens in the financial market.

Keywords: black-scholes equation, inverse fourier transform, financial options, bayesian estimation, R2OpenBUGS.

1. INTRODUCTION

A financial derivative may be defined as a financial instrument whose value depends on (or derived) from another asset or market to serve as a reference. The assets of which it depends is called the underlying asset. Very often, the variables underlying the derivatives are traded asset prices. Overall, it is a signed contract between two parties, buyer and seller, which define a future payment according to the behavior of the price of the asset. In some cases, the fair price of the derivative is independent of the model used to describe the behavior of assets over time and, in other cases, depending on the model, whose case will be discussed at this work.

Moreover, bearing in mind that speculation and coverage of the risk of a position of the active are underlying the main uses of derivatives, the area of risk management seeks to introduce models to control a fair price in the market. Moreover, risk management, inherent in the market has shown an exponential growth driven by the rapid development of information technology, which, in turn, has facilitated their operation and diversification. Thus, the bags which are negotiated and traded products provide alternative investment and greater coverage with more and better information (Hull, 2006) (Venegas, 2008). In 1900, L. Bachelier introduced a model involving Brownian motion (observed in nature by Brown in 1826) to model fluctuations of the Parisina bag.

However, thanks to the contributions of F. Black, M. Scholes (Black & Scholes, 1973) and RC Merton (Merton, 1973), winner of the Nobel Prize for Economics in 1997 for his outstanding contributions to financial mathematics in continuous time, it is possible to understand the estimate of the value of an option (or value of the premium). The model is known as the Black-Scholes equation; whose solution is the price of a European option (call or put) when the end condition is the intrinsic value of the option. Intrinsic value is the actual value of the option on the maturity date (expiry) of the contract. Furthermore, one of the advantages of the Black-Scholes model is that this partial differential equation is transformed into the heat diffusion equation, which has explicit solutions. Since then, the equation has become very popular because it represents the basis for valuing many diverse derivatives because, for different solutions, border represent the prices of many derivative products on the market (Venegas, 2008). Because of the developments in financial engineering and the significant role played by the study of the bag for better decision-making, it has become essential to use mathematical models that achieve response to the investor so that their results are representative of the reality.

In the first part of the work, theoretical aspects to consider along the article are described. After that deduction of the equation of the Black-Scholes, it is made from a stochastic differential equation for the evolution of the price of an active. To finally reach the solution using the separation of variables method and Fourier series.

2. MATERIALS AND METHODS

2.1 Derivation of the Black-Scholes Equation

2.1.1 Evolution of the price of a derivative

The Black-Scholes model consists of two assets: Bond (Bond) and the share price (Stock). The first evolves deterministically, and the second is random evolution, as described below.

The value of a European option's underlying asset is modeled by a stochastic process $\{S_t\}_{t\geq 0}$, solution of stochastic differential equation of the form

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

Where $\{W_t\}_{t\geq\mathbf{0}}$is a geometric Brownian motion, which incorporates elements of risk and uncertainty in the dynamics of that variable. It is assumed that the asset price

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that pays no dividends; follows a geometric Brownian motion, see (Neil, et to the, 2010). Considering that the variable S_t takes the value of zero, then the equation [1] we can write as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \qquad (2)$$

Therefore, S_t the price of the asset at a time t, μ the expected rate of return (or return medium) andσ the volatility of the asset price. We assume that, in this market, you can trade at any time, without costs.

In this context, we think that the average expected rate of return divided by the asset price is constant. Then, if S_t the asset's price at a time t, the *drift* parameter must be μS_t, for some constant μ. This means that in a short time interval Δt, the change would S_t be expected μS_t.

Now, if we consider that the volatility of the price of the asset is always zero, for a short period, [1], it has to

$$\Delta S_t = \mu S_t \Delta t$$

Then, for $\Delta t \to 0$

$$dS_t = \mu S_t dt$$

However, in practice, the volatility is not zero. If we think of the percent return $\frac{\Delta S_t}{S_t}$ in a time interval Δt, it is reasonable to assume that the variability of the value is the same, independent of the asset valueS_t. This means that the standard deviation of the change of the price of the asset, for the time interval Δt should be proportional to the price of the asset. I.e., that the model represents is given by

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t} \qquad (3)$$

Whichϵfollows a normal distribution $N(0,1)$, note that if $\Delta t \to 0$, then

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \qquad (4)$$

We are characterizing the geometric Brownian motion, which μ is the expected rate of return and σ the volatility of the asset price.

Note that in equation [4] Shows that the increases are divided $S_t$by variables with normal distribution standard for a small time mean $\mu\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$ for a small time Δt.

2.1.2 Black-Scholes equation

It is introducing the deduction and solution of the Black-Scholes equation for valuing European call and put options on stocks that do not pay dividends, which has been fundamental to the growth and success of financial engineering in the last 20 years (Hull, 2006) (Serrano, 1993), (Venegas, 2008).

As we saw in the previous section, assume that the asset price follows a geometric Brownian motion means:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad (5)$$

Where dW_t is a Wiener process.

IsV the current price of the option, so at the time t, the current value of the derivative is denoted by St. It is not necessary to specify whether it is a *call* or *put* option (buy or sell). To value options, we must develop some tools, such as the formula Ito which is a generalization of the chain rule of the usual calculation functions. The critical point is that it V is a function S_t and t this is it $V\ (S_t, t)$. Then, as $V\ (S_t, t)$ a class function then we C^2 have an example of a function for a random variable $\{S_t\}_{t\geq 0}$. Using Ito's lemma, we have,

$$dV = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}\right) dt + \frac{\partial V}{\partial S_t}\sigma S_t dW_t \qquad (6)$$

Shows that to have a stochastic equation for the price of the option $V\ (S_t, t)$, i.e., given the time t and the price of the current derivative still cannot obtain a single value for the option. To do this, we assume a portfolio formed by an option and a quantity $\Delta$of the underlying asset. The value of the portfolio over time t is

$$\pi(S_t, t) = V(S_t, t) - \Delta(S_t, t) S_t \qquad (7)$$

and the value of the portfolio between time t and $t + dt$ is given by

$$d\pi = dV - \Delta dS_t \qquad (8)$$

Note that the amount of the asset we own over time t does not change between time t and $t + dt$, the price of the asset changes by dS_t, therefore the option price changes by dV and consequently, the value of the portfolio changes by $d\pi$ (Neil, et to the, 2010), (Hull, 2006). Substituting [6] in [8] we have

$$d\pi = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}\right) dt + \frac{\partial V}{\partial S_t}\sigma S_t dW_t - \Delta dS_t \qquad (9)$$

Replacing [5] in [9], Further simplifying must be

$$d\pi = \sigma S_t\left(\frac{\partial V}{\partial S_t} - \Delta\right) dW_t + \left(\mu S_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \frac{\partial V}{\partial t} - \mu \Delta S_t\right) dt$$

The factor $\left(\frac{\partial V}{\partial S_t} - \Delta\right)$ is very important because it controls the stochastic element in the portfolio $d\pi$ and therefore the risk of it. If we take $\Delta = \frac{\partial V}{\partial S_t}$ for everything $t \in T$, we have indeed

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$$d\pi = \left(\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \frac{\partial V}{\partial t}\right) \mathrm{d}t \tag{10}$$

Which is an entirely deterministic equation for the value of the portfolio at each time t. An important consequence is that the risk was eliminated obtaining a portfolio of zero risks. Now, since the case had not acquired the underlying asset, and instead have chosen the option of risk where capital has been invested in a bank, in this case, our increasing portfolio value during the same period$t \rightarrow t + dt$, by a quantity

$$d\pi = r\pi dt \tag{11}$$

which r represents the rate of risk free rate, replacing [10] in [11] you have:

$$r\pi dt = \left(\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \frac{\partial V}{\partial t}\right) dt \tag{12}$$

In addition, if we substitute [7] in [12] is obtained

$$r\left[V - S_t \frac{\partial V}{\partial S_t}\right] = \left(\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \frac{\partial V}{\partial t}\right) \tag{13}$$

which it is valid for all times t during the life of the option, that is, [13] it is equivalent to

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S_t^2} = rV \tag{14}$$

This equation is known as the differential equation of Black-Scholes.

Interestingly, the expected return μ does not appear in the equation [14], this is because an essential property of the Black-Scholes equation and not involving any variable that is affected by the choice of the investor risk. The variable μ depends on the preference of risk, the more time is assumed risk by the investor, the higher the value μ required, see (Hull, 2006). Note that the only parameters involved in the model and should be identified are volatility σ and interest rate risk free r.

2.2 Solution Black-Scholes Equation

The differential equation is a Black-Scholes EDP second linear order; It is a second-order partial derivative with a second order. It is linear in the sense that the two functions are the solution of the equation, and then any linear combination of the two functions is the solution of the equation. On the other hand, to solve the equation [14], we have to meet the boundary conditions, which are defined below.

2.2.1 Boundary conditions for European options

We denoted by $V_c(S_t, t)$ and $V_p(S_t, t)$ the prices of European call and put options respectively, strike or exercise price K, maturity date Tat a time $t < T$ and S_t the current price of the asset.

- **European call option**

The price of a European call option $V_c(S_t, t)$ satisfies

$$\begin{cases} V_c(S_t, t) = max\{S_t - K, 0\} & for \quad 0 < S_t < \infty, \\ V_c(0, t) = 0 & for \quad 0 \leq t \leq T, \\ \lim\limits_{S_t \to \infty} V_c(0, t) = S_t - Ke^{-rt} & for \quad 0 \leq t \leq T. \end{cases}$$

The first condition is given to the definition of choice; that is, for an arbitrary t must verify that $S_t > K$ the option will be exercised and its value is given $S_t - K$, otherwise the option will not be exercised, and its value is zero, this condition is known as the end condition $T = t$. The second and third conditions are derived from the assumption that the asset price follows a geometric Brownian motion. If $S_t = 0$ for some time $t < T$ and the option $S_T = 0$ will not be exercised, thus obtaining the second condition (initial condition). Now, if $S_t \rightarrow \infty$ the value of the option approximates the value of the asset, that is, if S_t it becomes too large the option is exercised and its value is given by $S_t - K$, obtaining the third condition (condition at infinity).

- **European put option**

In the case of a *put* $V_p(S_t, t)$ option meets:

$$\begin{cases} \mathrm{V_P(S_t, t) = \max\{K - S_t, 0\}for} \quad 0 < \mathrm{S_t} < \infty, \\ \mathrm{V_P(0, t) = K} \quad \mathrm{for} \quad 0 \leq \mathrm{t} \leq \mathrm{T}, \\ \lim\limits_{\mathrm{S_t} \to \infty} \mathrm{V_P(0, t) = S_t - Ke^{-rt} for} \quad 0 \leq \mathrm{t} \leq \mathrm{T}, \end{cases}$$

Where the reasoning is analogous to that of European call options.

It is considered the following initial value problem and boundary for a European call option given by,

$$\begin{cases} \frac{\partial V_c}{\partial t} + rS_t \frac{\partial V_c}{\partial S_t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_c}{\partial S_t^2} = rV_c \\ V_c(S_t, t) = max\{S_t - K, 0\} for \quad 0 < S_t < \infty, \\ V_c(0, t) = 0 \quad for \quad 0 \leq t \leq T, \\ lim_{S_t \to \infty} V_c(0, t) = S_t - Ke^{-rt} for \quad 0 \leq t \leq T. \end{cases} \tag{15}$$

The first step is to make the coefficients of this equation constant dependent on the independent variable S_t, to perform the following substitution, thereby

$$\tau = \frac{1}{2}\sigma^2 t, \quad S_t = Ke^x \quad \text{and} \quad \mathrm{V_c(S_t, t)} = KC(x, \tau),$$

In fact, $x = \ln\left(\frac{S_t}{K}\right)$ considering the partial derivatives is has

$$\frac{\partial x}{\partial S_t} = \frac{1}{S_t}, \quad \frac{\partial x}{\partial t} = 0, \quad \frac{\partial \tau}{\partial S_t} = 0, \quad \frac{\partial \tau}{\partial t} = -\frac{1}{2}\sigma^2$$

To replace the new variables, using the chain rule, we obtain

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$\frac{\partial V_c}{\partial t} = -\frac{1}{2}\sigma^2 K \frac{\partial C}{\partial \tau}$, $\frac{\partial V_c}{\partial S_t} = e^{-x} K \frac{\partial C}{\partial x}$ and

$$\frac{\partial^2 V_c}{\partial S_t} = \frac{1}{K} e^{-2x} \left(\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} \right)$$

Substituting the partial derivatives in [15] has the following function in terms of the new function

$$\frac{\partial^2 C}{\partial x^2} + \frac{\partial C}{\partial x}\left(\frac{2r}{\sigma^2} - 1\right) - \frac{2r}{\sigma^2} C = \frac{\partial C}{\partial \tau}$$

Indeed, the final condition $(T = t)$ becomes an initial condition $(\tau = 0)$ of the form

$$V_c(S_t, t) = C(x, 0) = max\{e^x - 1\,, 0\}$$

and the boundary conditions (when $S_t \to 0$ and $S_t \to \infty$) are equivalent to

$$V_c(0,t) = V_c(K \lim_{x\to-\infty} e^x, t) = \lim_{x\to-\infty} C(x,\tau) = 0,$$
$$\lim_{S_t\to\infty} V_c(S_t, t) = \lim_{x\to\infty} KC(x,\tau) = \lim_{x\to\infty} C(x,\tau) = e^x - e^{-\tau\gamma}$$

where $\gamma = \frac{2r}{\sigma^2}$. Finally in terms of the system it $C(x,\tau)$ is given by

$$\begin{cases} \frac{\partial C}{\partial \tau} = \frac{\partial^2 C}{\partial x^2} + \frac{\partial C}{\partial x}(\gamma - 1) - \gamma C \\ C(x,\tau) = max\{e^x - 1\,, 0\} \quad for \quad 0 < x < \infty, \\ C(x,\tau) = \ 0 \quad for \quad 0 \le \tau \le T, \\ lim_{x\to\infty} C(x,\tau) = e^x - e^{-\tau\gamma} \quad for \quad 0 \le \tau \le T. \end{cases} \tag{16}$$

A convert the equation it Black-Scholes, in canonical form, the diffusion equation is obtained, to thereby realize a new change of variable, i.e.

$$C(x,\tau) = e^{\alpha x + \beta \tau} u(x,\tau)$$

Applying the chain rule to the new variable change and replacing the partial derivatives in [16] it is obtained

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (2\alpha + \gamma - 1)\frac{\partial u}{\partial x} + (\alpha^2 + \gamma\alpha - \alpha - \gamma - \beta)u \tag{17}$$

Then the term $\frac{\partial u}{\partial x}$ vanishes when

$$2\alpha + \gamma - 1 = 0 \tag{18}$$

and the term u is canceled when

$$\alpha^2 + \gamma\alpha - \alpha - \gamma - \beta = 0 \tag{19}$$

I.e., equations [17] and [18] have a unique solution when
$\alpha = -\frac{1}{2}(\gamma - 1)$ and $\beta = -\frac{1}{4}(\gamma + 1)^2$

If we replace α and β values [17], you can type the Black-Scholes equation in the form diffusive, i.e.

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

Therefore, we have the following system corresponding to the diffusion equation on a bounded domain with no initial conditions,

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad for \quad -\infty < x < \infty \\ u(x,0) = u_0(x); \\ \lim_{x\to-\infty} u(x,\tau) = 0 \\ \lim_{x\to-\infty} u(x,\tau) = \lim_{x\to\infty} \frac{e^x - e^{-\tau\gamma}}{e^{\left(\frac{1+\gamma}{2}\right)x} - e^{\left(\frac{\gamma-1}{2}\right)x}} \end{cases} \tag{20}$$

It is important to note that

$$u(x,0) = u_0(x) = \begin{cases} e^{\left(\frac{1+\gamma}{2}\right)x} - e^{\left(\frac{\gamma-1}{2}\right)x} & for \quad x > 0 \\ 0 & for \quad x \le 0 \end{cases}$$

To find the solution of the system [20] the separation of variables is used, i.e.,

$$u(x,\tau) = \phi(x)H(\tau)$$

The fundamental idea is reduced [20] to a system of ODE's. Indeed,

$$\frac{\partial u}{\partial \tau} = \phi(x)H'(x)\,, \qquad \frac{\partial^2 u}{\partial x^2} = \phi''(x)H(\tau),$$

Replacing the differential equation [20] we have,

$$\phi(x)H'(\tau) = \phi''(x)H(\tau) \tag{21}$$

Now if we perform the separation of variables and dividing [21] it $\phi(x)H(\tau)$ is obtained by

$$H^{-1}(\tau)H'(\tau) = \phi^{-1}(x)\phi''(x)$$

Equating to an arbitrary constant $-\lambda$ or also known as constant separation must be

$$H^{-1}(\tau)H'(\tau) = \phi^{-1}(x)\phi''(x) = -\lambda$$

Where two linear ordinary equations are obtained

$$\frac{1}{H(\tau)}H'(\tau) = -\lambda \quad \Leftrightarrow \quad H'(\tau) + \lambda H(\tau) = 0 \tag{22}$$

$$\frac{1}{\phi(x)}\phi''(x) = -\lambda \quad \Leftrightarrow \quad \phi''(x) + \lambda\phi(x) = 0 \tag{23}$$

Then, the solution [22] is given by

$$H(\tau) = ce^{-\lambda\tau}$$

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It c is an arbitrary constant. Now, to study the solution [23] as there are characteristic values λ for which $\phi(x)$ it has no trivial solution to this discuss the following cases.

Initially, the problem across the boundary line is

$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda\phi = 0 \\ |\phi(\pm\infty)| < \infty \end{cases} \tag{24}$$

- If $\lambda < 0$, then the roots of the characteristic equation $\rho^2 + \lambda = 0$ are real and so $\rho = \pm\sqrt{-\lambda}$ then

$$\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

Where c_1 and c_2 are arbitrary constants, now if we apply the boundary condition, the solution must be a linear combination of solutions and decrease exponentially growing, therefore not met $|\phi(\pm\infty)| < \infty$.

- If $\lambda = 0$, then $\frac{d^2\phi}{dx^2} = 0$ the solution is a characteristic value with constant characteristic function $\phi(x) = c_1$.

- If $\lambda > 0$, then the roots $\rho^2 + \lambda = 0$ are complex conjugates, in effect $\phi(x) = e^{\pm i\sqrt{\lambda}x}$ here have $\cos\sqrt{\lambda}x$ and $\sin\sqrt{\lambda}x$ are linearly independent solution $\frac{1}{\phi}\frac{d^2\phi}{dx^2}$ and indeed any solution the boundary problem is expressed as a linear combination of the two solutions given,

$$\phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

Note that the set of characteristic values is also known as the spectrum; in our case, we have a continuous spectrum $\lambda \geq 0$.
Finally, we get

$$u(x,\tau) = e^{-\lambda\tau}\left(c_1 \, cos\sqrt{\lambda}x + c_2 \, sin\sqrt{\lambda}x\right) \tag{25}$$

The generalized principle of superposition, see (Ioro, 1988) (Stanley, 1993) we can integrate [25] on the spectrum, i.e.,

$$u(x,\tau) = \int_0^\infty \left(c_1(\lambda)\, e^{-\lambda\tau}\cos\sqrt{\lambda}x + c_2(\lambda)e^{-\lambda\tau}\sin\sqrt{\lambda}x\right)\mathrm{d}\omega \tag{26}$$

Where $c_1(\lambda)$ and $c_2(\lambda)$ are arbitrary functions λ. Making the following change of variable $\lambda = \omega^2 \Leftrightarrow \mathrm{d}\lambda = 2\omega\mathrm{d}\omega$, we have $u(x,\tau)$ is given by

$$u(x,\tau) = \int_0^\infty \left(A(\omega)\cos\omega x e^{-\omega^2\tau} + B(\omega)\sin\omega x\, e^{-\omega^2\tau}\right)\mathrm{d}\omega$$

If we apply the initial condition $u(x,0) = u_0(x)$

$$u_0(x) = \int_0^\infty (A(\omega)\cos\omega x + B(\omega)\sin\omega x)\,\mathrm{d}\omega$$

Where $A(\omega)$ and $B(\omega)$ are arbitrary functions. Can now be expressed $u(x,\tau)$ in terms of complex exponential whose purpose is to take control of the solution as the real line, i.e.

$$u(x,\tau) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega x}\, e^{-\omega^2\tau}d\omega \tag{27}$$

Applying the initial condition $u(x,0) = u_0(x)$ again verified that,

$$u_0(x) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega x}\,\mathrm{d}\omega \tag{28}$$

Where we have [28] is a representation of the Fourier integral, and (Haberman, 1987). So $c(\omega)$ is the Fourier transform $u_0(x)$, i.e.,

$$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(x)e^{i\omega x}\,dx \tag{29}$$

Substituting $c(\omega)$ in [29] and since x it is a dummy variable, ente $x = s$, obtaining

$$u(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(s)\left[\int_{-\infty}^{\infty} e^{-\omega^2\tau}e^{i\omega(x-s)}d\omega\right]ds \tag{30}$$

If we make the following variable change and we $g(x) = \int_{-\infty}^{\infty} e^{-\omega^2\tau}e^{i\omega x}\mathrm{d}\omega$, like integrating [30] contains $g(x-s)$ ay not $g(x)$ because the interest is in finding the function $g(x)$ whose Fourier transform is $e^{-\omega^2\tau}$ and calculate $g(x-s)$. As $e^{-\omega^2\tau}$ a gaussian see (Haberman, 1987), we have

$$g(x) = \sqrt{\frac{\pi}{\tau}}\, e^{-\frac{x^2}{4\tau}}$$

Therefore, the solution is given by

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s)e^{-\frac{1}{4\tau}(x-s)^2}ds$$

Now, we see $u_0(s) = u_0(x)$ that $u_0(x)$ indeed as defined in [20] so

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\,\tau}}\int_0^\infty \left[e^{\left(\frac{1+\gamma}{2}\right)x} - e^{\left(\frac{\gamma-1}{2}\right)x}\right]e^{-\frac{1}{4\tau}(x-s)^2}ds$$

Therefore a way of expressing the previous integrating in a more resumed is if we do $y = \frac{(s-x)}{\sqrt{2\tau}}$, ie $s = x + y\sqrt{2\tau} \Leftrightarrow \mathrm{d}s = \sqrt{2\tau}\mathrm{d}y$, replacing and analyzing the limits of integration have

- If $s = 0$ then $y = -\frac{x}{\sqrt{2\tau}}$.
- If $s \to \infty$, then the upper limit is $\lim_{s\to\infty}\frac{(s-x)}{\sqrt{2\tau}} = +\infty$.

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then,

$$\mathrm{u}(\mathrm{x},\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\mathrm{x}}{\sqrt{2\tau}}}^{+\infty}\left[\mathrm{e}^{\left(\frac{1+\gamma}{2}\right)(\mathrm{x}+\mathrm{y}\sqrt{2\tau})}-\mathrm{e}^{\left(\frac{\gamma-1}{2}\right)(\mathrm{x}+\mathrm{y}\sqrt{2\tau})}\right]\mathrm{e}^{-\frac{1}{4\tau}\left(\mathrm{x}-(\mathrm{x}+\mathrm{y}\sqrt{2\tau})\right)^2}\mathrm{dy}$$

Note that the above integrating what can separate as two full-time to distribute, if we complete the first comprehensive squares get

$$\int_{-\frac{\mathrm{x}}{\sqrt{2\tau}}}^{+\infty}\mathrm{e}^{\left(\frac{1+\gamma}{2}\right)(\mathrm{x}+\mathrm{y}\sqrt{2\tau})}\,\mathrm{e}^{-\frac{1}{2}\mathrm{y}^2}\mathrm{dy}=\frac{\mathrm{e}^{\frac{1}{2}(\gamma+1)\mathrm{x}+\frac{1}{4}(\gamma+1)\sqrt{2\tau}}}{\sqrt{2\pi}}\int_{-\frac{\mathrm{x}}{\sqrt{2\tau}}}^{+\infty}\mathrm{e}^{-\frac{1}{2}\left(\mathrm{y}-\frac{(\gamma+1)\sqrt{2\tau}}{2}\right)^2}\mathrm{dy}$$

Now, if we do $z=y-\frac{(\gamma+1)\sqrt{2\tau}}{2}$ and if we study the new limits of integration have

- If $y=-\frac{x}{\sqrt{2\tau}}$, then,$z=-\frac{x}{\sqrt{2\tau}}-\frac{(\gamma+1)\sqrt{2\tau}}{2}$
- If $y\to\infty$, then, $\lim_{y\to\infty}z=+\infty$

That is,

$$\int_{-\frac{\mathrm{x}}{\sqrt{2\tau}}}^{+\infty}\mathrm{e}^{\left(\frac{1+\gamma}{2}\right)(\mathrm{x}+\mathrm{y}\sqrt{2\tau})}\,\mathrm{e}^{-\frac{1}{2}\mathrm{y}^2}\mathrm{dy}=\frac{\mathrm{e}^{\frac{1}{2}(\gamma+1)\mathrm{x}+\frac{1}{4}(\gamma+1)\sqrt{2\tau}}}{\sqrt{2\pi}}\int_{-\frac{\mathrm{x}}{\sqrt{2\tau}}-\frac{(\gamma+1)\sqrt{2\tau}}{2}}^{+\infty}\mathrm{e}^{-\frac{1}{2}(\mathrm{z})^2}\mathrm{dz}$$

Similarly, you can get the integral.

Furthermore, recall that the function of the normal distribution with zero mean and unit variance is given by

$$\mathrm{N}(\mathrm{d})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\mathrm{d}}\mathrm{e}^{-\frac{1}{2}\mathrm{s}^2}\mathrm{ds}$$

Therefore, if $-z=s$ we can express the solution as

$$\mathrm{u}(\mathrm{x},\tau)=\mathrm{e}^{\frac{1}{2}(\gamma+1)\mathrm{x}+\frac{1}{4}(\gamma+1)\sqrt{2\tau}}\mathrm{N}(\mathrm{d}_1)-\mathrm{e}^{\frac{1}{2}(\gamma-1)\mathrm{x}+\frac{1}{4}(\gamma-1)\sqrt{2\tau}}\mathrm{N}(\mathrm{d}_2)$$

Where,

$$d_1=\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(\gamma+1)\sqrt{2\tau}\text{and } d_2=\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(\gamma-1)\sqrt{2\tau}$$

Note that if we make changes variable in the opposite direction to return to the solution of the Black-Scholes equation we have

$$V_c(S_t,t)=S_t\,N(d_1)-Ke^{-r(T-t)}N(d_2) \qquad (31)$$

Where,

$$d_1=\frac{log\left(\frac{S_t}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\quad \text{y}\quad d_2=\frac{log\left(\frac{S_t}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

With $S_t>0$, $t\in(0,\infty]$ and using the fact that

$$\sqrt{2\tau}=\sqrt{\frac{2(\mathrm{T}-\mathrm{t})\sigma^2}{2}}=\sigma\sqrt{\mathrm{T}-\mathrm{t}}\quad\text{and}\quad\frac{\gamma+1}{2}=\frac{\mathrm{r}}{\sigma^2}+\frac{1}{2}.$$

Note that $N(d_1)$ and $N(d_2)$ they [31] represent the probability of exercising the option, plus agreement as defined d_1 and d_2 to the extent $S_t>K$ that $log\left(\frac{S_t}{K}\right)$ tends to infinity, therefore, d_1 and d_2 so will, otherwise $S_t<K$ tend to minus infinity, i.e. have the absolute certainty to exercise the option and not exercise, respectively.

Therefore, given the relationship known as *put-call* parity (Hull, 2006), (Venegas, 2008) you have to,

$$\mathrm{V_c}+\mathrm{Ke}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}=\mathrm{V_p}+\mathrm{S_t}$$

Through this relationship, given the value of a European *call* option, you can easily determine the value of European *put* option with the same aging time t and preset price K, and vice versa. It is essentially a *put*that is less risky than a *call*. A considerable variation of S, leaves the investor discovered a *call*, however, in much *put* to lose K.

In this case, for calculating a *call* option V_c it is necessary to calculate the value of its parameters σ, r, K, T, S_t. Since volatility σ is a measure of uncertainty, and it is potentially stochastic in the model estimation is performed using the classical and Bayesian approach, the other parameters are known from the financial market information.

2.3 Estimation of Volatility

2.3.1 Focus classic

Classical estimation is necessary to define the distribution of assigned behavior probability samples and determine the manner that best describes this information.

To this end, and considering $\phi=\sigma^2$ the random variable

$$R_t=\frac{S_t-S_{t-1}}{S_{t-1}}\sim N(0,\phi) \qquad (32)$$

Which defines the performance of an asset over time $t-1$ and time t (today). Consequently, the density function for R_t is given by

$$f(r|\phi)=\frac{1}{\sqrt{2\pi\phi}}exp\left(-\frac{r^2}{2\phi}\right),\quad \phi>0 \qquad (33)$$

To obtain an estimate ϕ used the maximum likelihood method which consists in finding $\hat{\phi}_{MV}$ a way that maximizes the likelihood function

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$$L(r_1,\cdots,r_n|\phi) = \prod_{i=1}^{n} f(r_i|\phi) = (2\pi\phi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\phi}\sum_{i=1}^{n} r_i^2\right) \quad (34)$$

With $r_1, r_2, \cdots, r_k$ a sample size n. Indeed, considering the function

$$\ln L(r_1,\cdots,r_n|\phi) = -\frac{n}{2}\ln(2\pi\phi) - \frac{1}{2\phi}\sum_{i=1}^{n} r_i^2 \quad (35)$$

Where is an unbiased estimator as follow?

$$\hat{\phi}_{MV} = \frac{1}{n-1}\sum_{i=1}^{n} r_i^2 \quad (36)$$

2.3.2 Bayesian approach

From the Bayesian approach, the parameter is considered as a random variable that can be modeled using a distribution continuous priori probability $h(\phi)$ which information, in terms of probabilities, can be updated by observations of a sample $r_1, r_2, \cdots, r_n$ as suggested (Bernardo et to the, 1993) and (Gelman et to the, 2003). Accordingly, a posterior distribution provides a comprehensive description of the random number ϕ obtained from the quantization of priors and the sample information is obtained.

To determine the density function of the posterior distribution, it is observed, of the Bayes Theorem that from likelihood function $L(r_1,\cdots,r_n|\phi)$ given in [34] and a prior distribution $h(\phi)$ it is possible to find the posterior distribution as follows:

$$h(\phi|r_1,\cdots,r_n) = \frac{L(r_1,\cdots,r_n|\phi)h(\phi)}{\int_0^{\infty} L(r_1,\cdots,r_n|\phi)h(\phi)} \quad (37)$$

Then as $\phi > 0$ the Inverse Gamma distribution and prior candidate distribution ofϕ, which allows asymmetry within its structure it is assumed. Thus the density function is defined as

$$h(\phi) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\phi^{-(\alpha+1)} \exp\left(-\frac{\beta}{\phi}\right), \quad \alpha, \beta > 0 \quad (38)$$

From the [37], posterior distribution is given by

$$h(\phi|r_1,\cdots,r_n) = h(\phi) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\phi^{-(\alpha_1+1)} \exp\left(-\frac{\beta_1}{\phi}\right) \quad (39)$$

Which $\alpha_1 = \alpha + \frac{n}{2}$ and $\beta_1 = \beta + \sum_{i=1}^{n}\frac{r_i^2}{2}$ coincides with the Gamma Reverse given [38], distribution parameters (α_1, β_1).

3. RESULTS AND DISCUSSIONS

The time series used are the daily returns of the shares of the company "Colombiana de Hidrocarburos" ECOPETROL SA to calculate the number of returns, given in [32]. It has been used closing prices daily of existing shares on the market in the period between 02 / January / 2018 and 15 / August / 2018, for a total of 151 observations obtained on the website www.bvc.com.co.

The histogram in Figure-1 shows the price series data does not follow a normal distribution, slightly skewed to the left and mesokurtic, with a coefficient of asymmetry -0.3361 and kurtosis -1.2615.

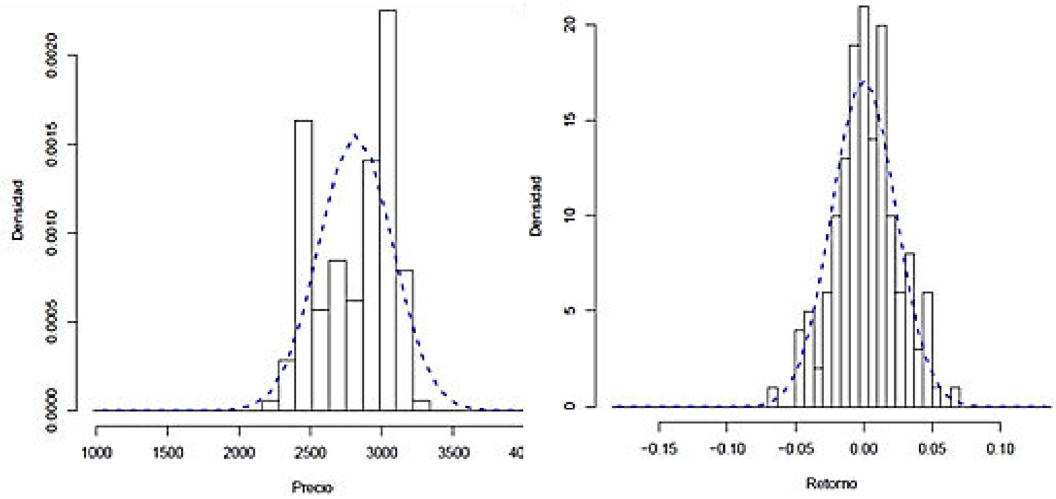

Figure-1. Histogram actions ECOPETROL.

In Figure-2. The differences were observed between the number of closing prices $\{P_t\}_{t=1}^{T=151}$ and the number of returns $\{R_t\}_{t=1}^{T=150}$, which behaves statistically, as a steady process, with a p-value of the 0.01 die by Dickey-Fuller a level of significance of the 5%.

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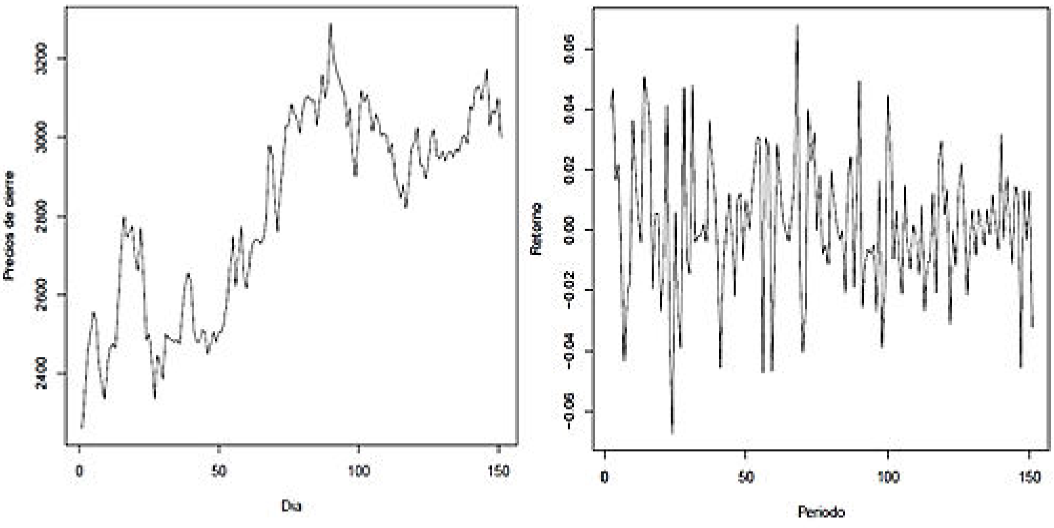

(a) Precios de cierre (b) Retornos

Figure-2. Series of prices and returns ECOPETROL.

Also, in Table-1 it is noted that the series R_t is similar to a normal distribution, since it is slightly skewed to the left and leptokurtic, with a coefficient of asymmetry -0.018 and kurtosis 0.1516. However, applying the Shapiro-Wilk and Jarque- Bera to determine whether the series R_t behaves like a normal distribution; indeed the p-values are 0.7022 and 0.8847, respectively, which there is sufficient evidence to establish that the series R_t does not follow a normal distribution with the significance level 5%.

Table-1. Descriptive actions ECOPETROL.

Measure	Statistical	
	Closing price	**Return**
Half	2816.689	0.00215853
Median	2895	0.0008
variance	64969.3	0.000541103
Des. Standard	254.8908	0.0232616
Asymmetry	-0.3361003	-0.0180576
kurtosis	-1.261524	0.152163

In Table-2 are observed values hyperparameter of Inverse Gamma prior distribution, which was evaluated using the statistical software R, and R2OpenBUGS worked with the library.

According to these measures shows that the distribution has a positive asymmetry, state natural distribution. In the case of hyperparameter is evident that the scale is smaller compared to the way it induces distribution more leptokurtic, i.e. assign a higher probability for low values of volatility.

Table-2. Measurements of the prior distribution.

Apriori		GammaInverse
hyperparameter	α_1	**3.0611**
	β_1	0.00066
Measurements	Half	0.0003
	Median	0.0002
	fashion	0.0002

In Figure-3. It is noted for posterior distribution σ^2 based on the prior distribution reference. In this case, it was evidenced that the posterior distribution has a symmetrical behavior within a range. Accordingly, the back half as a measure that summarizes the information as σ^2 shown in Table-3 is taken.

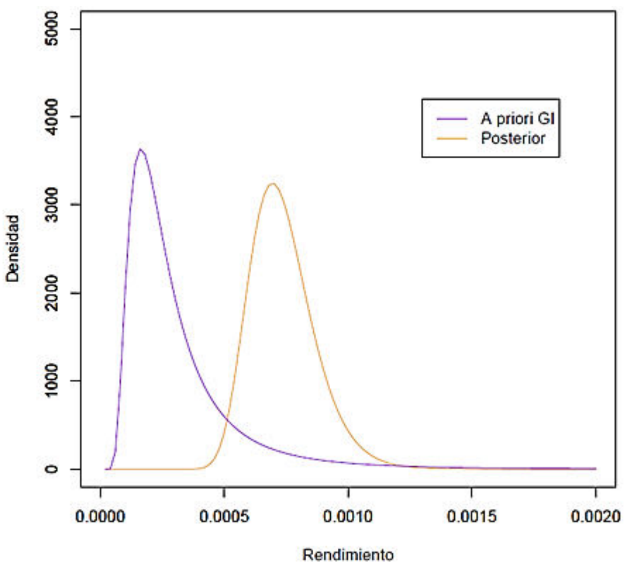

Figure-3. Form the prior distribution and subsequent.

Table-3. Dimensions of the posterior distribution.

Focus	$\widehat{\sigma}^2$	$\widehat{\sigma}$	Interval
Classic	0.00054	.0232	(0.0208, 0.0262)
Bayes	0.00059	0.0236	(0.0227, 0.0242)

Once determined the value of the volatility from the different approaches, the remaining parameters in the model [14] are taken freeform according to the behavior of the market.

How much is won or lost, really, for a European *call* option? Suppose you want to buy a European *call* option underlying asset for shares of Ecopetrol.

The contract establishes the first day that performs operations on the BVC; in our case in January 2018, the share price was $S_t = \$2260$on that day. The contract is set to one year,$T = 1$, with a strike price $K = \$2520$ also estimated that volatility is $\sigma_c = 0.0232$ (classic) and $\sigma_B = 0.0236$ (Bayes) and a free interest rate risk $r = 0.07214344608$. Indeed, the price to pay for every financial option according to the estimation of volatility from the classical and Bayesian approach is 1142.39 and 947.55, respectively. Check the expiration date of the option, the owner of this decides whether exercised or not such contract; he realizes that the share price is $ 2650.

4. CONCLUSIONS

In this work, the mathematical model was derived for the assessment of options known as the Black-Scholes equation, which consists of a linear partial differential equation of second order with initial and boundary conditions. The model allows us to find the courage to pay for exercising the right to a European call option (or put), taking as independent variables t and S_t overtime and that is the value of the asset over time t; known values of the parametersr rate risk free rate, σ volatility, K set price (strike) and T ripening time.

This deduction is made from a stochastic differential equation describing the price of a derivative with a random component and a deterministic. It is found that the value of the option is independent of the expected asset μ, which is one of the initial performance parameters.

As the model is linear possesses an analytical solution, which is obtained by transforming the equation of Black-Scholes is a diffusion equation which is solved by combining the methods of separation variables and inverse Fourier transform. Finally, from the parity ratio, you can be obtained the value of the put or (call) corresponding European option.

Finally, note that in [31] does not appear μ. We can also be replaced S_t and since this is a state variable since this is the asset's price at the time of the contract, it could be the average buying and selling. The r and σ parameters must be identified and T and K contained in the contract. Moreover, we can, knowing the value of the option at any intermediate time, i.e., it is an asset to be traded at any time. If we start to move these values, we noticed a sensitivity to$T$, since the price of the option increases if you increase the maturation time T. Also, there is a sensitivityσ to increases for the price if volatility rises.

However, using Bayesian methods, it was observed that the estimation of the parameter volatility tends to take a range of narrower values found in the classical approach, i.e., said regions credibility to 95% have lower variability in the estimate. Also, it was shown that Bayesian methods achieve better capture information returns than the classic method.

To conclude, if the share price differs gains or losses on options and then make the best financial decision depends on the value of K, that is, if $S_t \geq K$, where S_t is the price of the known future action, should exercise the option, It implies a profit by buying the underlying S_t-K. If you take into account what you pay for the option but take to future value T, then, it actually wins (or possibly lost) S_t-K-$V_c(S_t, t)e^{rT}$. On the other hand, if $K < S_t$, should not exercise the option, which implies a loss $V_c(S_t, t)e^{rT}$ by purchasing the contract. Accordingly, the evidence to evaluate the premium from the volatility estimate from the Bayesian approach presents a lower risk ratio.

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