NUMERICAL APPROXIMATION OF INDIRECT OPTIMAL CONTROL OF JEBBA HYDROELECTRIC POWER PLANT

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ABSTRACT

There has been unending research on the utilization of hydropower resources in Nigeria with several models proposed for an optimal generation. Of great importance among these models are those that are useful in the system management and controller design. This paper presents an optimal control problem that is formulated around a system model for the regulation of reservoir operating head at Jebba hydroelectric power plant. It also presents a means of solving a two-point boundary value problem resulting from an optimality equation. A conjugate gradient algorithm was adopted as an iterative numerical technique for the approximation of the optimal inflow required to ensure that the operating head follows a predefined trajectory. The result shows the feasibility of the control algorithm, its independence on the initial guess for the control and a relative error of 0.2% between the setpoint and the computed terminal operating head. The solution provides a means of optimal power generation on the cascaded Kainji-Jebbahydro power station and recommended in the realization of the physical controller.

Keywords: boundary value problem, conjugate gradient, numerical approximation, operating head, pontryagin.

1. INTRODUCTION

The availability of electricity on the Nigerian power grid depends greatly on the hydropower plant. Although the energy mix comprises of roughly 85% thermal power stationsbut most are faced with a shortage of fuel for the prime movers. The three major hydro stations contribute roughly15 %to the installed capacity and an average of ¼ of the available capacity in the year. This calls for the need to improve the performance of these hydropower stations for optimal operation[1] [2] [3].

Two of the hydropower plants are located on the river Niger, with the source from the Guinea Highlands. They are the Kainji hydroelectric power plant (KHEPS) on 09°51′45″ N, 04°36′48″ E and Jebba hydroelectric power plant (JHEPS) on 9°08′08″ N, 4°47′16″ E. The two stations are operated in cascade since they are located on the same river. The optimal operation of these stations in the presence of uncertain inflow, unit's availability, evaporation and frequency management-imposed constraints has been an issue for operators and researchers. While the KHEPS can have its reservoir operating head varies from 24m to 42m, JHEPS has a relatively constant operating head withing 25m to 29.2m and a nominal head of 26.1m. To ensure that the head at JHEPS is kept with the tight limits, operators rely on intuition and experience rather than a systemic approach. Hence the water is often spilled away to the downstream of JHEPS, otherwise, it evaporates at KHEPS and the security of generating resources is not certain [4] [5].

There have been different models proposed for optimal operation of the stations but most are based on the prediction and forecast of the stochastic inflow, they are useful for system study and management. A better model that can be useful in controller design should be based on system dynamics. Such is the model proposed by [6], it could predict the performance of the stations under varying inflows and generating unit availability.

To realize an optimal controller for regulating the operating head at JHEPS, an optimal control problem is formulated around the proposed dynamical model by [6]. Unfortunately, solutions to such a problem are usually challenging. The solutions can be intractable, computationally intensive or impossible to realize. In addition, there is no general technique to solve such a problem since each problem is unique in terms of the model, the functional and the constraints [7] [8].

Numerous authors had proposed strategies for taking care of an optimal control issue; these techniques can be as nonlinear programming for taking care of a limited dimensional control issue. Some of the known techniques include are the steepest descent, gradient projection, conjugate gradient, Shooting, dynamic programming, collocation, quasi-linearization and so forth [9] [10] [11]. There has been no ideal technique as everyone has its own favorable circumstances and weaknesses. An engineer must examine these techniques and determine the fitting one for a given control problem [12].

Along these lines, this paper presents a numerical approximation of indirect method of optimal control developed around a Conjugate Gradient Algorithm. Numerical approach is embraced because a solution by analytical means may not be feasible. The solution to an optimal control problem via Pontryagin's minimum principle does lead to finding methods of solving a boundary value problem. The method of the Conjugate gradient is a mathematical strategy for determining the minimum of a function and functional [13]. The strategy was employed to take care of an unconstrained optimal control issue by [14]. Afterward, it was modified to solving optimal control problems with a constraint forced on the state variables and the control variables [15] [16]. It added the fast attributes of the steepest descent strategy in

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iterating towards the final solution with extra adjustment in guaranteeing fast convergence.

The methodology for utilizing a conjugate gradient algorithm to tackle a problem rooted in optimal control is discussed in detail in [14] [17]. The approach is feasible provided the chosen cost functional is differentiable and a suitable line search technique is available, it will always yield solutions after a few iterations.

2. THE MATERIAL AND METHOD

The aim of solving optimal control is to estimate suitable control function that causes a plant to satisfy some physical differential and algebraic constraints and also minimizing a selected performance index. The vast methods of solutions have been classified into direct and indirect approaches. This paper uses the indirect approach as applied to the JHEPS [18] [19] [20].

2.1 Optimal Control of the Hydropower Operating Head

The indirect solution depends on Pontryagin's base rule that enlarges the customary analytics of a variational approach with an optimality criterionon the control variable, such that is typically not in the fundamental performance index [12]. It has an extra prerequisite that the performance criteria should incorporate the control signal, which is normally not identified with the state equation. Thus, the performance index incorporates punishments for any deviation away from the desired state and a predefined estimation of the control signal.

The system dynamical equation is described in details [6] [21]. Let $h(m)$ represents the operating head, $Q_i\ (m^3/s)$ stands for inflow into the reservoir while $q\ (m^3/s)$ is the inflow into the penstock. The total inflow loss is represented by $Q_L\ (m^3/s)$ and $Q_S\ (m^3/s)$ is the discharge through spillway. $A_1\ (m^2)$ and $A_2\ (m^2)$ are the effective surface area and area of the scroll casing respectively. $\rho\ (kg/m^3)$ represents the density of water, $g(m/s^2)$ is the gravitational acceleration while η denotes the conversion efficiency.

Given that the electrical power output is expressed as Equation (1) and reduced to equation (2) since η, ρ, A_2 and g are all constants.

$$P = \sqrt{2}\,\eta\rho A_2\, g^{3/2} h^{3/2} \tag{1}$$

$$P = \mathbf{\Lambda}(h); \tag{2}$$

where $\mathbf{\Lambda}$represents a scalar valued function.

The dynamical equation for the operating head is described by Equations (3) and (4)[6].

$$\frac{dh}{dt} = -\, n A_1{}^{-1} A_2 \sqrt{2gh} + A_1{}^{-1}(Q_i - Q_L - Q_s) \tag{3}$$

$$\dot{h}(t) = \mathbf{f}(h, u); \tag{4}$$

$$u(t) = Q_i - Q_L - Q_s$$

where f also represents a scalar-valued function.

The controller goal is to determine optimum inflow $u \epsilon U$ that forces the system described by $\dot{h} = f(h, u, t)$ to conform to a predefined optimal trajectories $h^*(t)$ and at the same time minimize certain specified performance indices described by Equation (5).

$$J = \min \int_{t_0}^{t_f} \left(\mathbf{K_1}\left(h - h(T)\right)^2 + \mathbf{K_2}\left(u - u(T)\right)^2\right) dt \tag{5}$$

$$J = \min \int_{t_0}^{t_f} \boldsymbol{\varphi_1}(h, u, t)\, dt; \tag{6}$$

Subject to:

$$\dot{h} = \mathbf{f}(h, u,\ t) \quad ;\ t_0 \le t \le t_f$$

$$h(t_0) = h_0 \tag{7}$$

$$h(t_f) = h(T) \tag{8}$$

Where$\boldsymbol{K_1}$and $\boldsymbol{K_2}$are scalar constants, $h(T)$ is the operating head at the final time, $u(T)$ is the maximum allowable inflow while $\boldsymbol{\varphi_1}$a function.

Based on the Pontryagin minimum principles, a Hamiltonian function $\boldsymbol{H}(h, u, \lambda, t)$ is defined as Equation (9) such that $\lambda(t)$ represents the Lagrange multiplier or the costate.

$$H = \varphi(h, u, t) + \lambda\, f(h, u, t) \tag{9}$$

The following optimality conditions and boundary conditions are formulated in equation (10) to (14):

$$\frac{\partial \mathbf{H}}{\partial h} - \frac{d}{dt}[\lambda] = 0 \tag{10}$$

$$\dot{\lambda} = -\frac{\partial \boldsymbol{H}}{\partial h} \tag{11}$$

$$\frac{\partial H}{\partial \lambda} = f(h, u, t) - \dot{h} = 0 \tag{12}$$

$$\dot{h} = \mathbf{f}(h, u, t) \tag{13}$$

$$\frac{\partial \mathbf{H}}{\partial u} = 0 \tag{14}$$

Relating equations (3) and (5) to the optimality conditions, results in equations (15) to (20), where:

$$H = K_1\left(h - h(T)\right)^2 + K_2\left(u - u(T)\right)^2 + \lambda\left(-n_j \delta \sqrt{h} + \mu\, u\right) \tag{15}$$

$$\dot{h}(t) = -n_j \delta \sqrt{h} + \mu\, u \tag{16}$$

$$\dot{\lambda}(t) = -[2K_1\left(h - h(T)\right) - \tfrac{1}{2}\lambda\, n_j \delta \tfrac{1}{\sqrt{h}}] \tag{17}$$

$$0 = 2K_2\left(u - u(T)\right) + \mu\lambda \tag{18}$$

$$\mathrm{h}(\mathrm{t}_0) = \mathrm{h}_0 \tag{19}$$

$$\lambda(\mathrm{t}_f) = 1 \tag{20}$$

From equation (18), $u(t)$ can be expressed as equation (21) and when substituted into equation (16) and (17), it results in two sets of differential equations (22) and (23) with split boundary conditions of equations.

$$u = -\frac{\mu\lambda}{2\boldsymbol{K_2}} + u(T) \tag{21}$$

$$\dot{h} = -n_j\delta\sqrt{h} - \mu\left(\frac{\mu\lambda}{2\boldsymbol{K_2}} + u(T)\right) \tag{22}$$

$$\dot{\lambda} = -2\mathrm{K}_{\mathbf{1}}\big(h - h(T)\big) + \frac{1}{2}\lambda\, n_j\delta\, \frac{1}{\sqrt{h}} \tag{23}$$

$$\mathrm{h}(\mathrm{t}_0) = \mathrm{h}_0$$
$$\mathrm{h}(\mathrm{t}_f) = \text{unknown}$$
$$\lambda(\mathrm{t}_0) = \text{unknown}$$
$$\lambda(\mathrm{t}_f) = 1$$

The problem become that of finding the solution to a two-point boundary-value problem where $h(t_0)$ and$\lambda(t_f)$ are specified.

2.2 A Conjugate Gradient Solution of Indirect Optimal Control

Numerous numerical techniques attempt to exploit the idea whereby the state equation is solved from $t_0 \rightarrow t_f$ and the co-state in reverse $t_f \rightarrow t_0$wipes out the ill condition that makes researchers avoid the method of shooting [22]. Assessment of the canonical equations (22) and (23), shows that for a given $u(t)$, it is conceivable to separate the state and the co-state equations while solving for the optimal control. In contrast to the shooting strategies, the conjugate gradient method is an iterative procedure which begins with underlying speculation for the control and does not require an estimate for the co-state as it would be settled from in reverse ($t_f \rightarrow t_0$). The complete control procedure is presented in the next section.

2.3 Procedure for a Conjugate Gradient Solution of Indirect Optimal Control

The procedure for the numerical computation of optimal control follows the steps below. It was implemented in Microsoft EXCEL VBA® environment to obtain the results presented in the next section.

Step 1: Set up the TPBVP as follows:

$$\mathrm{f}_1(\mathrm{h},\mathrm{u},\mathrm{t}) = \dot{\mathrm{h}} = -\mathrm{n_j}\delta \mathrm{h}^{(\mathrm{i})^{1/2}} + \mu\mathrm{u}^{(\mathrm{i})}$$
$$\mathrm{f}_2(\mathrm{h},\lambda,\mathrm{t}) = \dot{\lambda} = -[2\mathrm{K}_1\left(\mathrm{h}^{(\mathrm{i})} - \mathrm{h}(\mathrm{T})\right) - \frac{1}{2}\lambda^{(\mathrm{i})}\left(\mathrm{n_j}\delta\mathrm{h}^{(\mathrm{i})^{-1/2}}\right)]$$

Step 2: Given that $\mathrm{i} = 0$ and $\mathrm{h}^{(\mathrm{i})}(\mathrm{t}_0) = \mathrm{h}_0$

Step 3: Assume values for $\mathrm{u}^{(\mathrm{i})}_{(\mathrm{t})}$ from $\mathrm{t}_0 \rightarrow \mathrm{t}_f$.

Step 4: Numerically integrates $\mathrm{f}_1(\mathrm{h},\mathrm{u},\mathrm{t})$ forward from $\mathrm{t}_0 \rightarrow \mathrm{t}_f$ using the Adams Moulton technique. Store the trajectories $\mathrm{h}^{(\mathrm{i})}$.

Step 5: Using the computed$\mathrm{h}^{(\mathrm{i})}(\mathrm{t}_f)$ and $\lambda^{(\mathrm{i})}(\mathrm{t}_f) = 1$, numerically solve $\dot{\lambda}^{(\mathrm{i})}$ from $\mathrm{t}_f \rightarrow \mathrm{t}_0$ using the same AdamsMoulton technique.

Step 6: Compute the Gradient vector denoted by $\mathrm{g}^{(\mathrm{i})}_{(\mathrm{t})}$;

$$\mathrm{g}^{(\mathrm{i})} = \frac{\partial\mathbf{H}}{\partial\mathrm{u}^{(\mathrm{i})}} = 2\mathbf{K_u}\big(\mathrm{u}^{(\mathrm{i})} - \mathrm{u}(\mathrm{T})\big) + \mu\,\lambda^{(\mathrm{i})}$$

Step 7: Estimate the gradient parameter $\beta^{(\mathrm{i})}$;

$$\beta^{(\mathrm{i})} = \frac{\|\mathrm{g}^{(\mathrm{i})}\|^2}{\|\mathrm{g}^{(\mathrm{i}-1)}\|^2}\ ;$$
$$\beta^{(0)} = \left\|\mathrm{g}^{(0)}\right\|^2$$

Step 8: Compute the direction of search $\mathrm{s}(\mathrm{t})$;

$$\mathrm{s}^{(\mathrm{k})} = -\mathrm{g}^{(\mathrm{i})} + \beta^{(\mathrm{i})}\mathrm{s}^{(\mathrm{i}-1)};$$
$$\mathrm{s}^{(0)} = -\mathrm{g}^{(0)}$$

Step 9: Find the $\xi^{(\mathrm{i})}{}_{(\min)}$ by selecting three sets of Fibonacci numbers ξ_1,ξ_2, ξ_3, then compute

$$\mathrm{u}^{(\mathrm{i}+1)}_{\xi_1} = \mathrm{u}^{\mathrm{i}} + {\xi_1}^{\mathrm{i}}\mathrm{s}^{(\mathrm{i})} \text{ and } \mathrm{J}^{\mathrm{i}}_{(\xi_1)}$$
$$\mathrm{u}^{(\mathrm{i}+1)}_{\xi_2} = \mathrm{u}^{\mathrm{i}} + {\xi_2}^{\mathrm{i}}\mathrm{s}^{(\mathrm{i})} \text{ and } \mathrm{J}^{\mathrm{i}}_{(\xi_2)}$$
$$\mathrm{u}^{(\mathrm{i}+1)}_{\xi_3} = \mathrm{u}^{\mathrm{i}} + {\xi_3}^{\mathrm{i}}\mathrm{s}^{(\mathrm{i})} \text{ and } \mathrm{J}^{\mathrm{i}}_{(\xi_3)}$$

Step 10: Compute the coefficient c, band $\psi^{(\mathrm{i})}{}_{(\min)}$as follows;

$$\mathbf{c} = \frac{1}{(\xi_1\varsigma - \xi_3)}\left[\frac{\big(\mathrm{J}_{(\xi_1)} - \mathrm{J}_{(\xi_2)}\big)}{(\xi_1 - \xi_2)} - \frac{\big(\mathrm{J}_{(\xi_2)} - \mathrm{J}_{(\xi_3)}\big)}{(\xi_2 - \xi_3)}\right]$$
$$\mathrm{b} = \frac{\big(\mathrm{J}_{(\xi_1)} - \mathrm{J}_{(\xi_2)}\big)}{(\xi_1 - \xi_3)} - \frac{(\xi_1 + \xi_2)}{(\xi_1 - \xi_3)}\left[\frac{\big(\mathrm{J}_{(\xi_1)} - \mathrm{J}_{(\xi_2)}\big)}{(\xi_1 - \xi_2)} - \frac{\big(\mathrm{J}_{(\xi_2)} - \mathrm{J}_{(\xi_3)}\big)}{(\xi_2 - \xi_3)}\right]$$
$$\xi^{(\mathrm{i})}{}_{(\min)} = \xi^{(\mathrm{i})} = -\frac{\mathrm{b}}{2\mathrm{c}}$$
$$\mathrm{u}^{(\mathrm{i}+1)} = \mathrm{u}^{(\mathrm{i})} + \xi^{(\mathrm{i})}\mathrm{s}^{(\mathrm{i})}$$

Step 11: Check If;

$$\left\|J(u^{(i+1)}) - J(u^{(i)})\right\| \leq 10^{-a},$$

where a is a constant

If FALSE;
let $\mathrm{i} = \mathrm{i} + 1$ and
$\mathrm{u}^{(\mathrm{i})} = \mathrm{u}^{(\mathrm{i}+1)}$
Return to Step 4.
If TRUE;
Let$\mathrm{u}^* = \mathrm{u}^{(\mathrm{i}+1)}$ and
output h^*

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Step 12: End

3. RESULTS AND DISCUSSIONS

The procedure in steps 1 to 12 was implemented in an EXCEL VBA® and the results are presented in this section. The optimal control and the state trajectories were determined under different operating conditions. To identify the operating parameters, this naming format was used: (*Number of Units,*$h(t_0)$*, Time (days), penalty*). The format can be defined as follows: given *Number of Units* turbo-alternators unit in operation with an initial reservoir head of $h(t_0)\ m$, determine the optimal control needed to move the head to the desired operating head of $26.1\ m$ in *Time(in days)*, under the condition that there is no penalty on the maximum inflow $u(T)$. Since the result is iterative, note that the last iterations represent the response at the optimum time.

3.1 Operating Condition 1:(5, 25.8, 1, u(T) Unpenalized)

The operating condition 1 implies that there are 5 units in operation with $h(t_0) = 25.8$. Find $u(t)$needed to raise the head to the $26.1m$ in 1 day, given that there is no penalty on under the condition that there is no penalty on $u(T)$. Figure1presents the head trajectories after each iteration. The algorithm converges at the 5thiterations, but the trajectories are not satisfactory. Apart from iterations1 that was diverging, no other iterations yielded a result with operating head close to $26.1\ m$.

Figure-2 shows the control input at each iteration. None of these controls is acceptable since they do not produce the desired operating head. Also $u(t_0) = 4710.59\ m^3/s$may too large and may not be available for release from KHEPS.

From the fundamentals of optimal control, the performance of the conjugate gradient solution must be revealed by the plot of the co-state (Figure-3), the gradient vector over time (Figure-4) and the convergence condition and the performance index after each iteration shown in Figure-5. The costate after every at each iteration must end in unity, the conjugate gradient vector must decrease with time, the performance index and the stopping criteria must also show decreasing characteristics.

Since the operating head does not conform to the predefined trajectory, there must be a penalty on the control. This is the case presented under the next subsection.

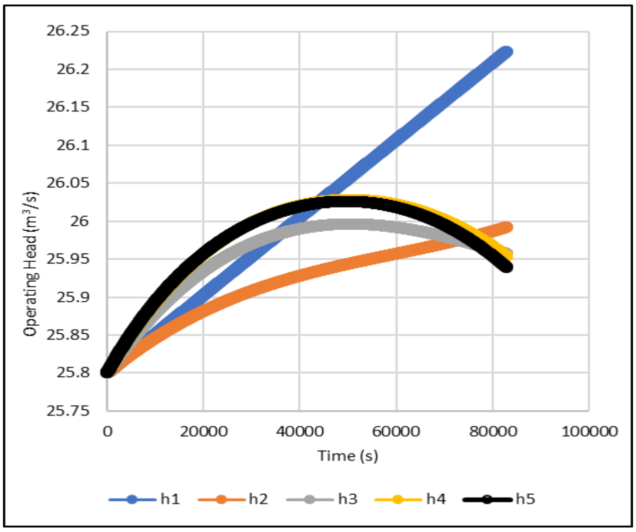

Figure-1. Head trajectory (5, 25.8, 1, u(T) Unpenalized).

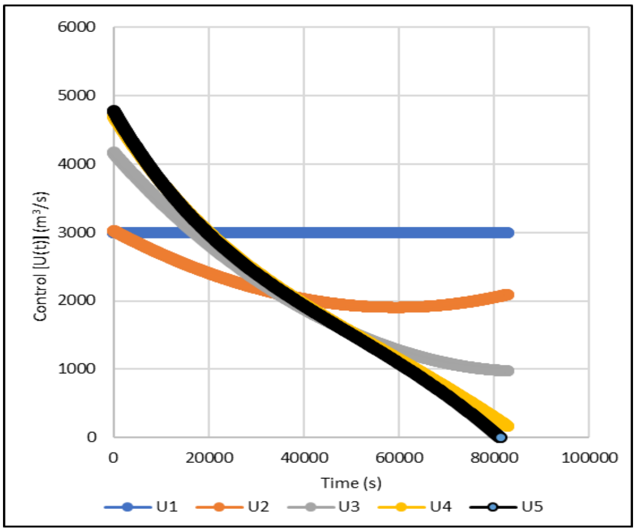

Figure-2. Control (5, 25.8, 1, u(T) Unpenalized).

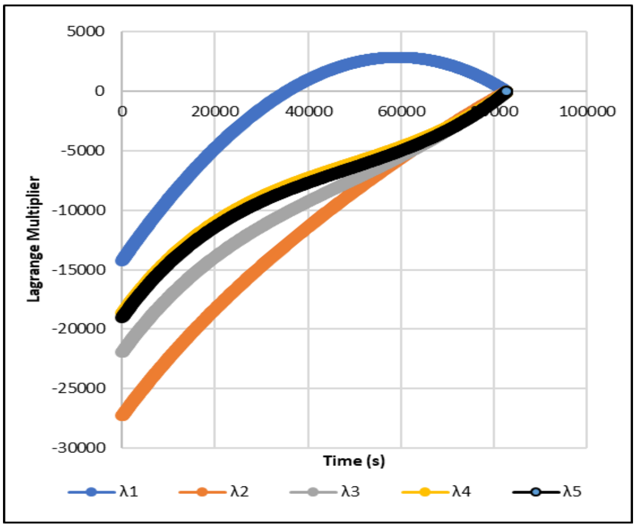

Figure-3. Lagrange multiplier (5, 25.8, 1, u(T) Unpenalized).

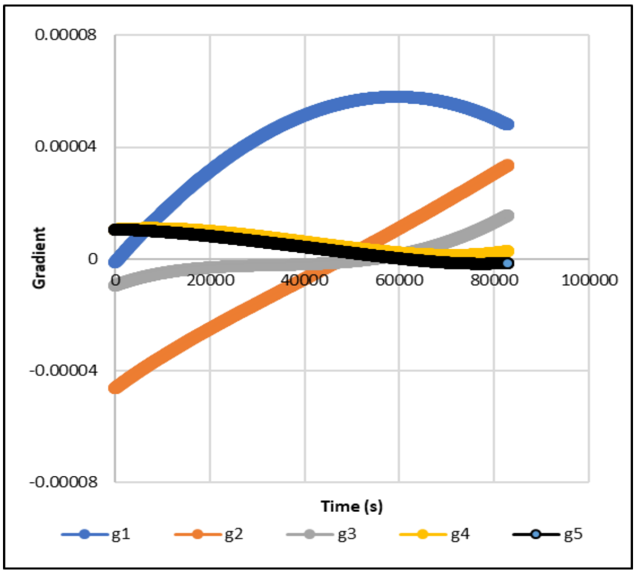

Figure-4. Gradient vector $(5, 25.8, 1, u(T)$ Unpenalized$)$.

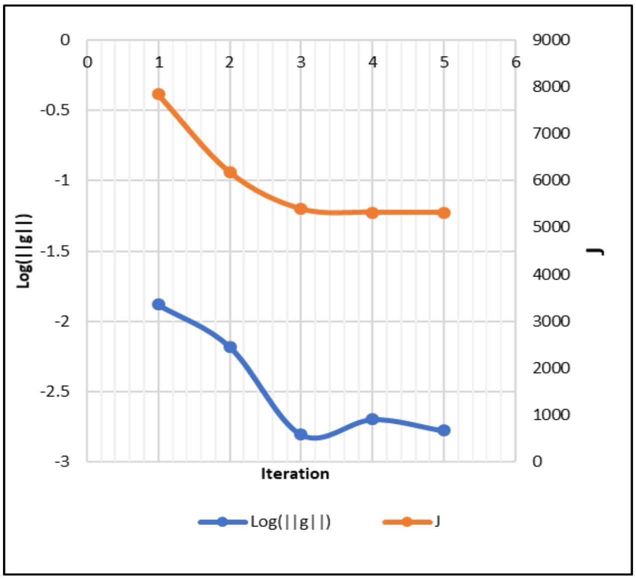

Figure-5. Performance index and gradient against number of iterations$(5, 25.8, 1, U(T)$ Unpenalized$)$.

3.2 Operating Condition 2:$(5, 25.8, 1, u(T) = 1800\ m^3/s)$ and $u(t_0) = 2500\ m^3/s$

In operating condition 2, $u(t_0)$ is not allowed to be lower than $2500\ m^3/s$while $u(T)$ cannot exceed$1800\ m^3/s$. Figure-2 shows that the optimal head trajectory on deviates from $26.1m$ with a relative error of around 0.2%. This is a better result as compared to the condition earlier considered. The desired optimal control that produces an expected trajectory is shown in Figure7. In addition, by imposing a constraint on $u(t_0)$ and $u(T)$may increase the convergence time while the slight overshoot in the state trajectory of Figure-1 can also be eliminated.

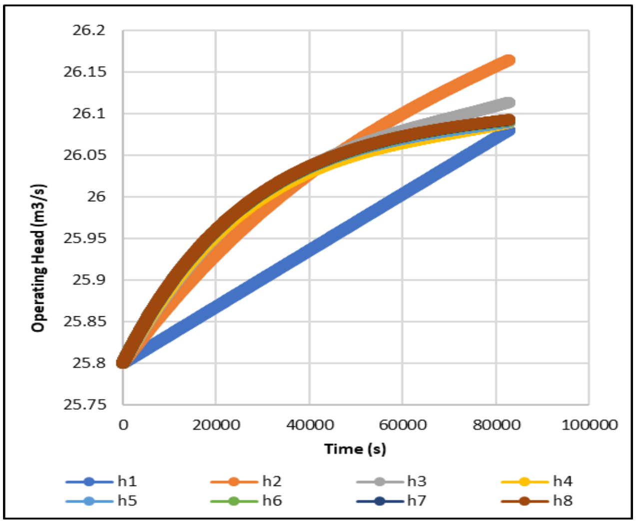

Figure-6. Head versus time $(5, 25.8, 1, u(T) = 1800\ m^3/s)$.

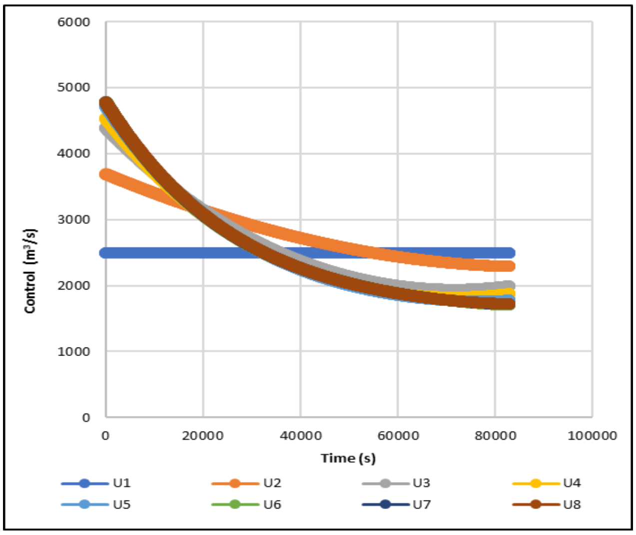

Figure-7. Control versus time $(5, 25.8, 1, u(T) = 1800\ m^3/s)$.

3.3 Operating Condition 3:$(5, 25.8, 1, u(T) = 1800\ m^3/s)$ and $u(t_0) = 500\ m^3/s$

The shooting method is highly sensitive to the initial guess but the conjugate gradient-based algorithm shows a unique difference. Here, the initial guess of $u(t_0) = 2500\ m^3/s$ under operating condition 2 decreases to $u(t_0) = 500\ m^3/s$. A similar trajectory to that of Figure-7 is presented in Figure-8 shows that the algorithm doesn't diverge provided that the initial guess is not outrageous or ambiguous.

Since the goal is to compute the required control, the resulting optimal control is presented in Figure-9. The control at t_f remain around the specified value of $1800\ m^3/s$.

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3.4 Operating Condition 4: $(5, 25.5, 3, u(T) = 2000\ m^3/s)$

It is also possible to consider a case where $h(t_0) = 25.5m$ and an optimal control is desired to raise this head to $26.1m$. If this is to be achieved in 1 day, the initial inflow required may not be available. Hence the time span is increased to $t_0 \rightarrow t_f 3$ days ($36\ hr$). Figure-10 reveals that the algorithm is still capable of estimating the desired trajectory with an estimated optimal control of Figure-11. The number of iterations required for the algorithm to converge is not far from those previously considered while $u(T)$ remain around 2000 m^3/s

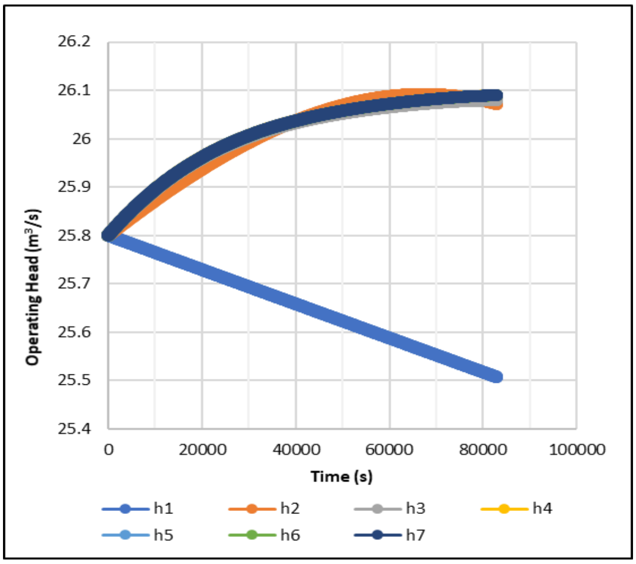

Figure-8. Head trajectory$(5, 25.8, 1, u(t_0) = 500\ m^3/s)$.

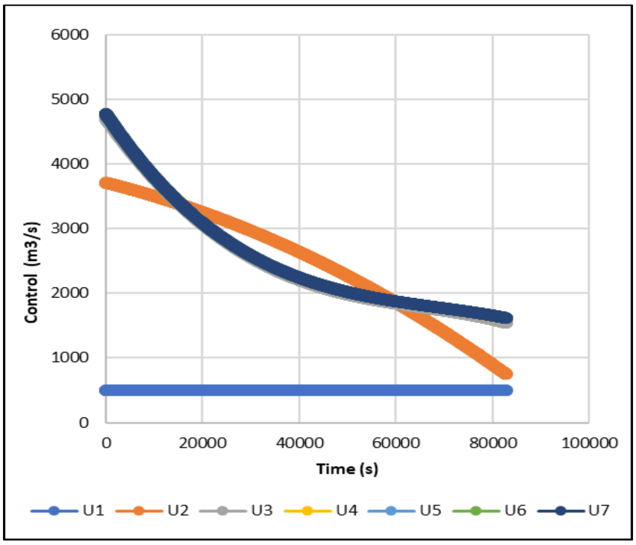

Figure-9. Control $(5, 25.8, 1, u(T)$ Penalized $\&u(t_0) = 500\ m^3/s)$.

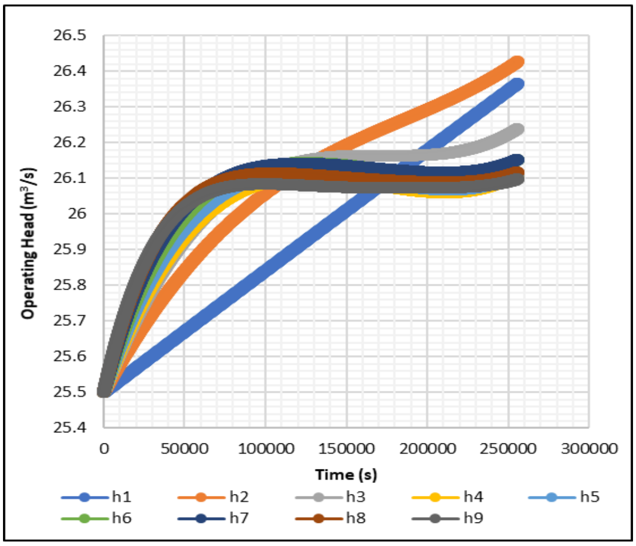

Figure-10. Head versus time $(5, 25.5, 3, u(T)$ Penalized$)$.

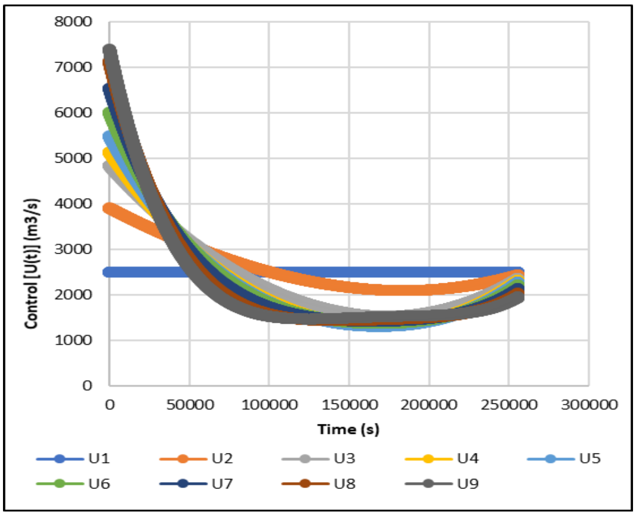

Figure-11. Control $(5, 25.5, 3, U(T)\ Penalized)$.

4. CONCLUSIONS

A suitable iterative algorithm for the estimation of optimal control for Jebba hydropower planthead was considered. The control algorithm works by approximation of the optimal control form a resulting two-point boundary value problem. Adams-Moulton numerical procedure supported with a Runge-Kuttaas the starter solves the decoupled states and costate differential equations while a conjugate gradient algorithm approximates the optimal control. Results presented show the feasibility of the control algorithm and the superiority to the shooting technique, since it does not deviate with an arbitrary initial guess. The algorithm is applicable irrespective of the time interval for which an optimal control is desired. The technique is recommended for use by the operators in the management of the station and in the development of a physical controller for the optimal generation from the cascaded JHEPS and KHEPS.

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