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VARIATIONAL RITZ-KANTOROVICH-EULER LAGRANGE METHOD FOR THE ELASTIC BUCKLING ANALYSIS OF FULLY CLAMPED KIRCHHOFF THIN PLATE

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ABSTRACT

The variational Ritz-Kantorovich-Euler Lagrange method is used in this article for solving the elastic buckling problems of rectangular thin plates ($2a \times 2b$) with fully clamped edges. Two cases were considered: biaxial uniform compression and uniaxial uniform compression of the plate. The plate material is assumed to be homogeneous and isotropic. Kantorovich approach is used in constructing the unknown deflection $u_z(x, y)$ as the product of an unknown deflection $f(x)$ in the x direction, and a basis function in the y direction derived from symmetry and clamped boundary conditions at $y = \pm b$. The variational problem over the domain then simplified to a variational problem over the one dimensional domain $-a \le x \le a$. Euler-Lagrange differential equation was used for extremization and the resulting fourth order Euler-Lagrange equation was solved to obtain the general expression for $f(x)$. Symmetry and implementation of the clamped conditions at $x = \pm a$ were used to obtain the eigenequation in each considered case as a transcendental equation. The eigenequations were solved to obtain the eigenvalues from which the critical buckling loads were determined. The eigenfunctions were found by solving the eigenvalue eigenvector problem. The results obtained were exact results as the variational problem was minimized in the entire domain and the clamped boundary conditions were identically satisfied. The results agree with previous reported results in the literature, validating the method employed.

Keywords: Ritz-Kantorovich-Euler Lagrange method, eigenequation, eigenfunction, critical buckling load, total potential energy functional, elastic buckling problem.

INTRODUCTION

Flat plates are a class of thin-walled structures (shells) with zero curvature. They are common in various engineering applications across civil, mechanical, aeronautical, naval and aerospace engineering fields [1-8]. Their peculiar features of high flexural stiffness, efficiency and strength to weight ratios recommend them to structural designs that optimize material requirements [1-8].

The foundations of the theory of plates were laid by the ground breaking studies presented by Kirchhoff and Love who formulated the classical linear plate theory (CPT). Since the inauguration of the CPT several other plate theories have been derived and developed to take care of the inadequacies of the CPT. Some of them are Reissner's sixth order plate theory [9], Mindlin's first order shear deformation plate theory, von Karman's plate theory, Shimpi's refined plate theory [10], Levinson's plate theory, Reddy's plate theory [5, 11].

Plate theories are constructed using

a) strain-displacement relations

b) displacement fields often derived from the kinematic relations, and

c) stress-strain relations [12-15].

The use of nonlinear strain-displacement relations enables the formulation of plate theories for large deformation problems since in that case the strains are nonlinear functions of displacement. The incorporation of the constitutive relations for elastic-plastic, visco-elastic, visco-plastic and piezoelectric, and other materials enable the derivation of plate theories for elastic-plastic, visco-elastic, visco-plastic and piezoelectric materials.

Plates are grouped into three broad classes depending on the ratio of the least inplane dimension (a) (characteristic inplane dimension) to the thickness (t) [12-15]

a) Thick plates are plates with $a/t \le 8$ to 10

b) Membrane plates are plates with $a/t \ge 80$ to 100

c) Thin plates are plates with 8 to $10 \le a/t \le 80$ to 100.

The buckling analysis of plates is a classical theme and numerous research studies have been conducted on elastic and inelastic buckling of plates of various shapes, sizes and material characteristics [12-16]. However, most of the available research reported on buckling of rectangular thin plates due to compression or shear loads assume uniformly distributed inplane loads applied at the edges. This assumption of uniform distribution of inplane loads is often unrealistic in practice because the rectangular plate buckling problem is usually the idealisation of a small part of the much larger and more complicated built-up plated structure and the loads applied are often exerted on the small portion by the adjoining structural elements. It is thus not easy to determine the real distribution of the loads on the boundary edges of the small rectangular part of the plate

under consideration because it requires modelling of the entire built-up plated structure [16].

However, the assumption of uniform distribution of inplane edge loads in the rectangular thin plate buckling problem is a convenient starting point. An illustrative case of non-uniform loading of plates that have been studied is the plate under two collinear point loads in compression.

Timoshenko and Gere [3] presented a simplistic energy based formulation and analysis of the buckling problem of rectangular plate under two collinear point compressive loads wherein the plane stress solution was completely obviated. Leissa and Ayoub [4] employed the finite element methodology to the studies of the rectangular thin plate buckling problem under two collinear point compressive loads. Jana and Bhaskar [17] have used trigonometric series methods to study the buckling of rectangular plates under various edge inplane load distributions in uniaxial compression. They presented some standard closed-form solutions for rectangular plates with simply supported edges. They found from their results, that the critical elastic buckling load is affected very significantly by the kind of load distribution along the edge.

Other studies of the behaviours of rectangular plates under nonuniform axial compressive inplane loads were presented by Christian [18] and Bert and Davarakonda, [19].

The focus of this paper is on the elastic stability problems of fully clamped rectangular thin plates due to (i) biaxial uniform compression and (ii) uniaxial uniform compression. Elastic buckling problems of fully clamped rectangular thin plates have not been extensively investigated and there are few published papers on the subject.

Elastic buckling problems of rectangular thin plates with two opposite edges clamped and the remaining two opposite edges simply supported have been investigated by Onah *et al* [20] using the single finite Fourier sine integral transform method, and by Onyia *et al* [21] using the Galerkin-Kantorovich method. Onyia *et al* [22, 23] obtained elastic buckling solutions for rectangular thin plates simply supported on two opposite edges, clamped and free respectively along the other two remaining edges using the simply finite Fourier single integral transform method and the Galerkin-Vlasov variational method respectively.

Ullah *et al* [24] used the finite integral transformation method to develop new mathematical solutions to the buckling problem of moderately thick clamped rectangular plates. Ullah *et al* [25] used the two dimensional generalized finite integral transformation technique to derive new mathematical solutions for the shear buckling problem of rectangular thin plates with fully clamped edges.

Ibearugbulem and Ezeh [26] studied the stability of axially compressed clamped thin rectangular plate using the Ritz method. Ezeh *et al* [27] used the Galerkin indirect variational method for the elastic stability investigations of fully clamped rectangular thin plates. Ibearugbulem [8] applied the Ritz direct variational method to investigate the stability of rectangular thin flat plates under uniaxial compression for various edge support conditions.

Levy [28, 29] presented solutions for the flexure and buckling of rectangular thin plates with clamped edges. Hwang and Lee [30] studied the stability of transversely isotropic plates subjected to various inplane loads at the edge for various boundary conditions. Bouazza et al [31] investigated the elastic stability of rectangular thin plates under uniaxial and biaxial compressive loads. Mandel and Mishra [32] used the finite element method to investigate the stability of rectangular plate under inplane loads. Wang *et al* [33] used the symplectic superposition method to derive new closed form solutions for the elastic buckling of rectangular thin plates for various boundary conditions.

Other significant research publications that highlight the recent contributions in the theme of plate theory in dynamics, flexure and buckling are Jamali *et al* [34], Eipakchi and Moshir [35], Orak and Salehi [36], Khoram *et al* [37], Javidi and Zand [38] and Zargaripoor *et al* [39].

In this work, the elastic buckling problems of rectangular thin plate under biaxial uniform and uniaxial uniform compressive loads are solved in a systematic and methodical manner using the variational Ritz-Kantorovich-Euler Lagrange method. The motivation in choosing the adopted methodology is to obtain closed form solutions for the eigenvalue problem that satisfy the governing domain equation and all the boundary conditions.

Theoretical Framework

Two basic methods are used to formulate problems of elastic buckling. They are:

a) differential equations of elastic stability methods, and

b) variational methods.

Differential Equation of Elastic Stability

The differential equation of elastic stability of thin plates assumed to be homogeneous, isotropic, linear elastic and subjected to inplane compressive forces, N_x, N_y and shear force N_{xy} applied at the plate edges is given by:

$$D\nabla^4 u_z + N_x \frac{\partial^2 u_z}{\partial x^2} + N_y \frac{\partial^2 u_z}{\partial y^2} + 2N_{xy} \frac{\partial^2 u_z}{\partial x \partial y} = 0 \qquad \dots (1)$$

where

$$\nabla^4 = \nabla^2 \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2$$
$$= \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \qquad (2)$$

∇^2 is the Laplacian and ∇^4 is the biharmonic operator

$u_z(x, y)$ is the deflection

x,y are the inplane Cartesian coordinates of the plate
D is the flexural rigidity of the plate.

$$D = \frac{Et^3}{12(1-\mu^2)} \tag{3}$$

E is the Young's modulus of elasticity of the material of the plate
t is the thickness of the plate
μ is the Poisson's ratio of the material of the plate.

The elastic buckling equation is solved over the plate domain, subject to boundary conditions on $u_z(x,y)$ which are defined by the conditions of edge supports.

Variational Methods

Variational methods of elastic stability analysis of plates rely on the minimization of the total potential energy functional of the plate. The total potential energy functional, Π, of a rectangular thin plate subjected to inplane compressive forces N_x, N_y and shear force N_{xy} applied at the plate edges $x = \pm a,\ \ y = \pm b$ where the origin is assumed at the centre of the plate and the plate material is assumed to be homogeneous, isotropic and linear elastic is expressed by:

$$\Pi = \frac{D}{2}\int_{-a}^{a}\int_{-b}^{b}(\nabla^2 u_z)^2 + 2(1-\mu)\left[\left(\frac{\partial^2 u_z}{\partial x \partial y}\right)^2 - \frac{\partial^2 u_z}{\partial x^2}\frac{\partial^2 u_z}{\partial y^2}\right]dxdy - \frac{1}{2}\int_{-a}^{a}\int_{-b}^{b}\left\{N_x\left(\frac{\partial u_z}{\partial x}\right)^2 + 2N_{xy}\frac{\partial u_z}{\partial y}\frac{\partial u_z}{\partial x} + N_y\left(\frac{\partial u_z}{\partial y}\right)^2\right\}dxdy \tag{4}$$

$$\Pi = U + V \tag{5}$$

U is the strain energy functional for bending; V is the potential energy functional of the loads applied to the thin rectangular plate.

$$U = \frac{D}{2}\int_{-a}^{a}\int_{-b}^{b}\left\{(\nabla^2 u_z)^2 + 2(1-\mu)((u_{z,xy})^2 - u_{z,xx}u_{z,yy})\right\}dxdy \tag{6}$$

$$V = -\frac{1}{2}\int_{-a}^{a}\int_{-b}^{b}\left\{N_x(u_{z,x})^2 + 2N_{xy}u_{z,y}u_{z,x} + N_y(u_{z,y})^2\right\}dxdy \tag{7}$$

where $u_{z,x} = \frac{\partial u_z}{\partial x},\ \ u_{z,y} = \frac{\partial u_z}{\partial y},\ \ u_{z,xy} = \frac{\partial^2 u_z}{\partial x \partial y},$

$u_{z,xx} = \frac{\partial^2 u_z}{\partial x^2},\ \ u_{z,yy} = \frac{\partial^2 u_z}{\partial y^2}$

METHODOLOGY

The problems we seek to solve in this paper are the elastic buckling problems of rectangular thin plates clamped along the four edges $x = \pm a,\ \ y = \pm b$ as shown in Figure-1 for the case of uniaxial uniform buckling and also shown in Figure-2 for the case of uniaxial uniform buckling. The origin of Cartesian coordinates is taken at the plate centre, in each considered case.

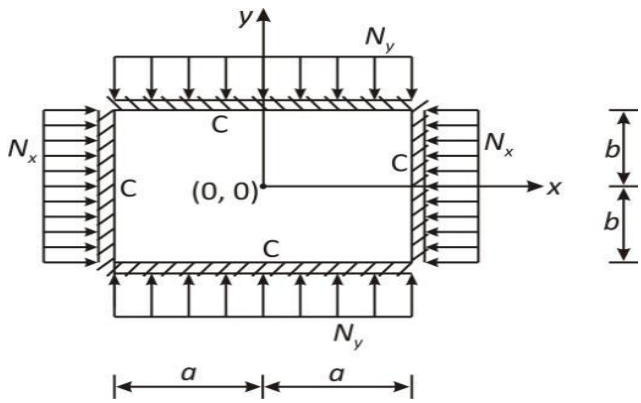


Figure-1. Rectangular thin plate with clamped edges with the edges $x = \pm a$ subject to compressive forces N_x and edges $y = \pm b$ subject to compressive force N_y.

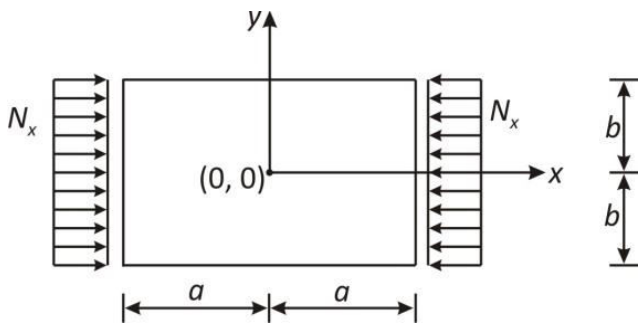


Figure-2. Rectangular thin plate $(2a \times 2b)$ with clamped edges $x = \pm a,\ \ y = \pm b$ where the edges $x = \pm a$ are subject to uniform uniaxial compressive load N_x and edges $y = \pm b$ are free of loads.

Variational Presentation of the Biaxial Elastic Buckling Problem

In the study the biaxial buckling problem considered is the case where $N_x = N_y = N$. The biaxial compressive force is the same in both the x and y coordinate directions. The total potential energy functional we seek to minimize then becomes:

$$\Pi = \frac{D}{2}\int_{-a}^{a}\int_{-b}^{b}\left\{(\nabla^2 u_z)^2 - 2(1-\mu)(u_{z,xx}u_{z,yy} - u_{z,xy}^2)\right\}dxdy - \frac{N}{2}\int_{-a}^{a}\int_{-b}^{b}(u_{z,x}^2 + u_{z,y}^2)\,dxdy \tag{8}$$

Kantorovich Assumption of the Displacement $u_z(x, y)$

Following Kantorovich's approach, the deflection $u_z(x, y)$ is chosen in the coordinate variable separable form as the product of the two functions $f(x)$ and $g(y)$ where $f(x)$ is a function of the x coordinate variable only; and $g(y)$ is a function that depends on the y coordinate variable alone.

Thus, $u_z(x, y) = f(x)g(y)$ (9)

The boundary conditions along the clamped edges $y = \pm b$ are:

$$u_z(x, y = b) = u_z(x, y = -b) = 0 \quad (10)$$

$$\frac{\partial u_z}{\partial y}(x, y = b) = \frac{\partial u_z}{\partial y}(x, y = -b) = 0 \quad (11)$$

Hence it follows that

$$g(y = b) = g(y = -b) = 0 \quad (12)$$

$$\frac{dg}{dy}(y = b) = g'(y = b) = \frac{\partial g}{\partial y}(y = -b)$$
$$= g'(y = -b) = 0 \quad (13)$$

Derivation of Coordinate Basis Function in the *y* Direction

The coordinate basis (shape) function in the y coordinate defined over the region $-b \le y \le b$; such that the boundary conditions are satisfied can be found by assuming $g(y)$ in the form of a polynomial of fourth degree as follows:

$$g_1(y) = e_1 y^4 + e_2 y^3 + e_3 y^2 + e_4 y + e_5 \quad (14)$$

where e_1, e_2, e_3, e_4 and e_5 are the polynomial constants.

Then, $g_1'(y) = 4e_1 y^3 + 3e_2 y^2 + 2e_3 y + e_4$ (15)

For symmetry of $g_1(y)$, $g_1(y = b) = g_1(y = -b)$
Then, $e_2 = e_4 = 0$ (16)

Then, $g_1(y) = e_1 y^4 + e_3 y^2 + e_5$ (17)

Differentiating with respect to y, we have:

$$g_1'(y) = 4e_1 y^3 + 2e_3 y \quad (18)$$

Applying the boundary conditions give:

$$g_1(y = b) = e_1 b^4 + e_3 b^2 + e_5 = 0 \quad (19)$$

$$g_1(y = -b) = e_1 b^4 + e_3 b^2 + e_5 = 0 \quad (20)$$

$$g_1'(y = b) = 4e_1 b^3 + 2e_3 b = 0 \quad (21)$$

$$g_1'(y = -b) = -4e_1 b^3 - 2e_3 b = 0 \quad (22)$$

Solving, $e_1 = -\frac{e_3}{2b^2}$ (23)

$$e_3 = -2b^2 e_1 \quad (24)$$

Then, $g_1(y) = e_1 y^4 - 2b^2 e_1 y^2 + e_5$ (25)

$$g_1(y = b) = e_1 b^4 - 2b^4 e_1 + e_5 = 0 \quad (26)$$

Solving, $e_5 = b^4 e_1$ (27)

Then, $g_1(y) = e_1 y^4 - 2b^2 e_1 y^2 + b^4 e_1$ (28)

$$g_1(y) = e_1(y^4 - 2b^2 y^2 + b^4) \quad (29)$$

$$g_1(y) = e_1(y^2 - b^2)^2 = e_1 g(y) \quad (30)$$

The coordinate basis function $g(y)$ is then found as:

$$g(y) = (y^2 - b^2)^2 \quad (31)$$

$g(y) = (y^2 - b^2)^2$ satisfies all the boundary conditions along the clamped edges $y = \pm b$.

Hence, the deflection is found by the Kantorovich procedure as:

$$u_z(x, y) = f(x)(y^2 - b^2)^2 \quad (32)$$

Ritz Total Potential energy Functional for the Elastic Stability Problem

We then seek to find $f(x)$ that minimizes the total potential energy functionalΠ. By differentiation,

$$u_{z,x} = f'(x)g(y)$$
$$u_{z,xx} = f''(x)g(y)$$
$$u_{z,y} = f(x)g'(y)$$
$$u_{z,yy} = f(x)g''(y)$$
$$u_{z,xy} = f'(x)g'(y)$$

We define a modified Ritz functional Π_* as follows:

$$\Pi_* = \frac{2\Pi}{D} = \int_{-a}^{a}\int_{-b}^{b} \left\{(\nabla^2 u_z)\right\}^2 - 2(1-\mu)(u_{z,xx}u_{z,yy} - (u_{z,xy})^2) - \alpha^2(u_{z,x}^2 + u_{z,y}^2)\right\} dxdy \quad (33)$$

where $\alpha^2 = \frac{N}{D}$ (34)

Substituting the expression for u_z into Equation (33), the modified Ritz functional becomes:

$$\Pi_* = \int_{-a}^{a}\int_{-b}^{b} \left\{(f''(x)g(y) + f(x)g''(y))^2 - 2(1-\mu)(f''(x)g(y)f(x)g''(y) - (f'(x)g'(y))^2) - \alpha^2((f'(x)g(y))^2 + (f(x)g'(y))^2)\right\} dxdy \quad (35)$$

where the primes denote space derivatives.
Hence,

$$f'(x) = \frac{df(x)}{dx},\ f''(x) = \frac{d^2 f(x)}{dx^2},\ g'(y) = \frac{dg(y)}{dy},$$
$$g''(y) = \frac{d^2 g(y)}{dy^2}$$

Simplifying, Π_* becomes:

$$\Pi_* = \int_{-a}^{a}\int_{-b}^{b} \left\{(f''(x)^2(g(y))^2 + 2f''(x)f(x)g''(y)g(y) + (f(x))^2(g''(y))^2 - 2(1-\mu)(f''(x)f(x)g''(y)g(y) - (f'(x))^2(g'(y))^2) - \alpha^2((f'(x)^2(g(y))^2 + (f(x))^2(g'(y))^2\right\} dxdy \quad (36)$$

Further simplification of Π_* gives:

$$\Pi_* = \int_{-a}^{a} \left\{(f''(x))^2 I_1 + 2f''(x)f(x)I_2 + (f(x))^2 I_3 - 2(1-\mu)(f''(x)f(x)I_2 - (f'(x))^2 I_4) - \alpha^2((f'(x))^2 I_1 + (f(x))^2 I_4)\right\} dx \quad (37)$$

in which I_1, I_2, I_3, I_4 are integrals defined as follows:

$$I_1 = \int_{-b}^{b} (g(y))^2 dy \quad (38)$$

$$I_2 = \int_{-b}^{b} g''(y)g(y)dy \quad (39)$$

$$I_3 = \int_{-b}^{b} (g''(y))^2 dy \quad (40)$$

$$I_4 = \int_{-b}^{b} (g'(y))^2 dy \quad (41)$$

In general, Π_* is expressed as:

$$\Pi_* = \int_{-a}^{a} F(x, f(x), f'(x)f''(x))dx \quad (42)$$

where $F(x, f(x), f'(x)f''(x))$ is the integrand.

$$F(x, f(x), f'(x), f''(x)) = (f''(x))^2 I_1 + 2f''(x)f(x)I_2 + (f(x))^2 I_3 - 2(1-\mu)(f''(x)f(x)I_2 - (f'(x)^2 I_4) - \alpha^2((f'(x))^2 I_1 + (f(x))^2 I_4) \quad (43)$$

Minimizing the Modified Ritz Functional with Respect to *f(x)*

The modified variational Ritz functional is minimized with respect to the unknown function *f*(*x*) if the Euler-Lagrange differential equation is satisfied. The Euler-Lagrange differential equation is given for the problem by:

$$\frac{\partial F}{\partial f(x)} - \frac{d}{dx}\frac{\partial F}{\partial f'(x)} + \frac{d^2}{dx^2}\frac{\partial F}{\partial f''(x)} = 0 \quad (44)$$

From Equation (43) for *F*, the partial derivatives of *F* in Equation (44) are evaluated as:

$$\frac{\partial F}{\partial f(x)} = 2f''(x)I_2 + 2f(x)I_3 - 2(1-\mu)f''(x)I_2 - \alpha^2 2f(x)I_4 \quad (45)$$

$$\frac{\partial F}{\partial f'(x)} = -2(1-\mu)(-2f'(x)I_4) - \alpha^2 2f'(x)I_1 = 4(1-\mu)f'(x)I_4 - 2\alpha^2 f'(x)I_1 \quad (46)$$

$$\frac{\partial F}{\partial f''(x)} = 2f''(x)I_1 + 2f(x)I_2 - 2(1-\mu)f(x)I_2 \quad (47)$$

Then the Euler-Lagrange differential equation becomes:

$$2f''(x)I_2 + 2f(x)I_3 - 2(1-\mu)f''(x)I_2 -$$

$$2\alpha^2 f(x)I_4 - \frac{d}{dx}(4(1-\mu)f'(x)I_4 - 2\alpha^2 f'(x)I_1) + \frac{d^2}{dx^2}(2f''(x)I_1 + 2f(x)I_2 - 2(1-\mu)f(x)I_2) = 0 \quad (48)$$

Hence,

$$2f''(x)I_2 + 2f(x)I_3 - 2(1-\mu)f''(x)I_2 - 2\alpha^2 f(x)I_4 - 4(1-\mu)f''(x)I_4 + 2\alpha^2 f''(x)I_1 + 2f^{iv}(x)I_1 + 2f''(x)I_2 - 2(1-\mu)f''(x)I_2 = 0 \quad (49)$$

We evaluate the integrals I_1, I_2, I_3 and I_4.

$$g(y) = (y^2 - b^2)^2 = y^4 - 2b^2y^2 + b^4$$
$$g'(y) = 4y^3 - 4b^2y$$
$$g''(y) = 12y^2 - 4b^2$$

$$I_1 = \int_{-b}^{b} (g(y))^2 dy = \int_{-b}^{b} (y^2 - b^2)^4 dy = \frac{256b^9}{315} \quad (50)$$

$$I_2 = \int_{-b}^{b} g''(y)g(y)dy = \int_{-b}^{b} (12y^2 - 4b^2)(y^2 - b^2)^2 dy = -\frac{256b^7}{105} \quad (51)$$

$$I_3 = \int_{-b}^{b} (g''(y))^2 dy = \int_{-b}^{b} (12y^2 - 4b^2)^2 dy = 25.6b^5 \quad (52)$$

$$I_4 = \int_{-b}^{b} (g'(y))^2 dy = \int_{-b}^{b} (4(y^3 - b^2y))^2 dy = \frac{256b^7}{105} \quad (53)$$

The Euler-Lagrange differential equation of elastic stability becomes explicitly:

$$2f''(x)\left(\frac{-256b^7}{105}\right) + 2f(x)25.6b^5 - 2(1-\mu)f''(x)\left(\frac{-256b^7}{105}\right) - 2\alpha^2 f(x)\frac{256b^7}{105} - 4(1-\mu)f''(x)\frac{256b^7}{105} + 2\alpha^2 f''(x)\frac{256b^9}{315} + 2f^{iv}(x)\frac{256b^9}{315} + 2f''(x)\left(\frac{-256b^7}{105}\right) - 2(1-\mu)f''(x)\left(\frac{-256b^7}{105}\right) = 0 \quad (54)$$

Simplification gives the explicit form of the Euler-Lagrange differential equation of elastic stability (in biaxial case) as the fourth order homogeneous ordinary differential equation (ODE). The ODE is linear.

$$b^4 f^{iv}(x) + ((\alpha b)^2 - 6)b^2 f''(x) - (3(\alpha b)^2 - 31.5)f(x) = 0 \quad (55)$$

Elastic Buckling of Clamped Rectangular Thin Plates under Uniaxial Uniform Compression

For the case of uniaxial buckling, the total potential energy functional is:

$$\Pi_* = \frac{D}{2}\int_{-a}^{a}\int_{-b}^{b}\{(\nabla^2 u_z)^2 - 2(1-\mu)(u_{z,xx}u_{z,yy} - u_{z,xy}^2)\}dxdy - \frac{N}{2}\int_{-a}^{a}\int_{-b}^{b} u_{z,x}^2 dxdy \quad (56)$$

and

$$\Pi_* = \int_{-a}^{a}\int_{-b}^{b}\{(\nabla^2 u_z)^2 - 2(1-\mu)(u_{z,xx}u_{z,yy} - u_{z,xy}^2) - \alpha^2 u_{z,x}^2\}dxdy \quad (57)$$

Using the expression for u_z we have:

$$\Pi_* = \int_{-a}^{a}\{(f''(x))^2(g(y))^2 + 2f''(x)f(x)g''(y)g(y) + (f(x))^2(g''(y))^2 - 2(1-\mu)(f''(x)f(x)g''(y)g(y) - (f'(x)^2(g'(y))^2) - \alpha^2(f'(x))^2(g'(y))^2\}dx \quad (58)$$

$$\Pi_* = \int_{-a}^{a}\{(f''(x))^2 I_1 + 2f''(x)f(x)I_2 + (f(x))^2 I_3 - 2(1-\mu)(f''(x)f(x)I_2 - (f'(x))^2 I_4) - \alpha^2(f'(x))^2 I_1\}dx \quad (59)$$

$$.\Pi_* = \int_{-a}^{a} F(x, f(x), f'(x), f''(x))dx\ . \quad (60)$$

$$F(x, f(x), f'(x), f''(x)) = (f''(x))^2 I_1 + 2f''(x)f(x)I_2 + (f(x))^2 I_3 - 2(1-\mu)(f''(x)f(x)I_2 - (f'(x))^2 I_4) - \alpha^2(f'(x))^2 I_1 \quad (61)$$

The Euler-Lagrange differential equation for this case of elastic buckling by uniaxial uniform compression is:

$$2f''(x)I_2 + 2f(x)I_3 - 2(1-\mu)f''(x)I_2 - \frac{d}{dx}(4(1-\mu)f'(x)I_4 - \alpha^2 2f'(x)I_1) +$$

$$\frac{d^2}{dx^2}(2f''(x)I_1 + 2f(x)I_2 - 2(1-\mu)f(x)I_2) = 0 \quad (62)$$

Simplifying, we have:

$$b^4 f^{iv}(x) + (\alpha^2 b^2 - 6)b^2 f''(x) + 31.5 f(x) = 0 \quad (63)$$

RESULTS

Solution of the Euler-Lagrange Differential Equation for Elastic Buckling under Biaxial Uniform Compression

The fourth order Euler-Lagrange differential equation of elastic stability for biaxial buckling under uniform compression in both the x and y coordinate directions of the rectangular plate which is obtained as Equation (55) is solved to obtain a general solution for $f(x)$ that minimizes the total potential energy functionalΠ.

Mathematical methods for solving ODEs can be used. Hence, general solutions can be obtained using differential operator (D-operator) methods, trial function methods, variation of parameters methods etc.

In this work, we used the method of trial functions to obtain a general solution for $f(x)$. Let the unknown function $f(x)$ be assumed in the form of the exponential function given by

$$f(x) = \exp sx = e^{sx}$$

where s is an undetermined and unknown parameter of the function $f(x)$ which we desire to find.

Then by differentiations, $f'(x) = se^{sx}$, $f''(x) = s^2 e^{sx}$, $f'''(x) = s^3 e^{sx}$ and $f^{iv}(x) = s^4 e^{sx}$

The ODE becomes:

$$b^4 s^4 e^{sx} + ((\alpha b)^2 - 6)b^2 s^2 e^{sx} - (3(\alpha b)^2 - 31.5)e^{sx} = 0 \quad (64)$$

Rearranging, we have:

$$[b^4 s^4 + b^2 s^2((\alpha b)^2 - 6) - (3(\alpha b)^2 - 31.5))e^{sx} = 0 \quad (65)$$

For nontrivial solutions of Equation (65) we require that e^{sx} must not vanish $(e^{sx} \neq 0)$.

Hence, the only condition for Equation (65) to be true and for the assumed trial solution to be a solution of the ODE is that the coefficient of e^{sx} in the Equation (65) must vanish.

This gives the auxiliary or characteristic equation as the polynomial expressed by:

$$b^4 s^4 + ((\alpha b)^2 - 6)b^2 s^2 - (3(\alpha b)^2 - 31.5) = 0 \quad (66)$$

Equation (66) can be considered a quadratic equation in $(bs)^2$. Considering Equation (66) as a quadratic equation in $(bs)^2$, the quadratic formula is used to obtain the two roots as:

$$b^2 s^2 = \frac{1}{2}\{-((\alpha b)^2 - 6) \pm \sqrt{((\alpha b)^2 - 6)^2 + 4(3(\alpha b)^2 - 31.5)}\} \quad (67)$$

Hence,

$$s^2 = \frac{1}{2b^2}\{-((\alpha b)^2 - 6) \pm \sqrt{((\alpha b)^2 - 6)^2 + 4(3(\alpha b)^2 - 31.5)}\} \quad (68)$$

Three possible types of roots exist for s^2 namely: (i) the roots are multiple (double) roots; (ii) the roots are real and distinct and (iii) the roots are complex conjugates.

The roots are multiple when the terms in the square root vanish. This corresponds to the condition for the general quadratic equation given by:

$$ax^2 + bx + c = 0$$

That is, $b^2 - 4ac = 0$ in $x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$ and $x = \frac{-b}{2a}$ (twice)

Hence for this problem,

$$((\alpha b)^2 - 6)^2 + 4(3(\alpha b)^2 - 31.5) = 0 \quad (69)$$

$$(\alpha b)^4 - 12(\alpha b)^2 + 36 + 12(\alpha b)^2 - 126 = 0 \quad (70)$$

Solving, $(\alpha b)^4 - 90 = 0$ (71)

$$\alpha b = \sqrt[4]{90} = 3.080070288 \quad (72)$$

Other possibilities for the roots of the equation are found by further simplification of Equation (68). Equation (68) can be expressed further as:

$$s^2 = \frac{1}{b^2}\left\{\frac{-((\alpha b)^2 - 6)}{2} \pm \frac{1}{2}\sqrt{((\alpha b)^2 - 6)^2 + 4(3(\alpha b)^2 - 31.5)}\right\} \quad (73)$$

or, $$s^2 = \frac{1}{b^2}\left\{-\left(\frac{(\alpha b)^2}{2} - 3\right) \pm \sqrt{\frac{((\alpha b)^2 - 6)^2 + 4(3(\alpha b)^2 - 31.5)}{2^2}}\right\} \quad (74)$$

$$s^2 = \frac{1}{b^2}\left\{-\left(\frac{(\alpha b)^2}{2} - 3\right)\right.$$

$$\pm\sqrt{\left(\frac{(\alpha b)^2}{2}-3\right)^2+(3(\alpha b)^2-31.5)}\Bigg\} \quad (75)$$

$$s^2=\frac{1}{b^2}\left\{\pm\sqrt{\left(\frac{(\alpha b)^2}{2}-3\right)^2+(3(\alpha b)^2-31.5)}-\left(\frac{(\alpha b)^2}{2}-3\right)\right\} \quad (76)$$

For positive sign, in Equation (76) we have

$$s^2=\frac{1}{b^2}\left\{\sqrt{\left(\frac{(\alpha b)^2}{2}-3\right)^2+(3(\alpha b)^2-31.5)}-\left(\frac{(\alpha b)^2}{2}-3\right)\right\} \quad (77)$$

Then, $s=\pm\lambda$ (78)

The two roots are:

$$s_1=+\lambda \quad (79)$$

$$s_2=-\lambda \quad (80)$$

where

$$\lambda=\frac{1}{b}\left\{\sqrt{\left(\frac{(\alpha b)^2}{2}-3\right)^2+(3(\alpha b)^2-31.5)}-\left(\frac{(\alpha b)^2}{2}-3\right)\right\}^{1/2} \quad (81)$$

For negative sign, in Equation (76) we have:

$$s^2=\frac{1}{b^2}\left\{-\sqrt{\left(\frac{(\alpha b)^2}{2}-3\right)^2+(3(\alpha b)^2-31.5)}-\left(\frac{(\alpha b)^2}{2}-3\right)\right\} \quad (82)$$

Re-arranging,

$$s^2=\frac{-1}{b^2}\left\{\sqrt{\left(\frac{(\alpha b)^2}{2}-3\right)^2+(3(\alpha b)^2-31.5)}+\left(\frac{(\alpha b)^2}{2}-3\right)\right\} \quad (83)$$

$$s^2=-\beta^2 \quad (84)$$

Hence, $s=\pm i\beta$ (85)

and $s_3=+i\beta$ (86)

$$s_4=-i\beta \quad (87)$$

where $\beta=\frac{1}{b}\left\{\sqrt{\left(\frac{(\alpha b)^2}{2}-3\right)^2+(3(\alpha b)^2-31.5)}+\left(\frac{(\alpha b)^2}{2}-3\right)\right\}^{1/2}$ (88)

General Solution for *f*(*x*)

The basis of linearly independent solutions for *f*(*x*), are thus:

$\exp(\lambda x)$, $\exp(-\lambda x)$, $\exp(i\beta x)$ and $\exp(-i\beta x)$

The general solution is then found by the linearity principle or the principle of superposition as:

$$f(x)=\bar{c}_1e^{\lambda x}+\bar{c}_2e^{-\lambda x}+\bar{c}_3e^{i\beta x}+\bar{c}_4e^{-i\beta x} \quad (89)$$

where $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$ and $\bar{c}_4$ are integration constants.

The general solution for *f*(*x*) in terms of the hyperbolic and trigonometric functions, are obtained by using the Euler formula and the relationships between hyperbolic and exponential functions:

$$e^{i\beta x}=\cos\beta x+i\sin\beta x \quad (90)$$

$$e^{-i\beta x}=\cos\beta x-i\sin\beta x \quad (91)$$

$$\sinh\lambda x=\frac{e^{\lambda x}-e^{-\lambda x}}{2} \quad (92)$$

$$\cosh\lambda x=\frac{e^{\lambda x}+e^{-\lambda x}}{2} \quad (93)$$

Then $f(x)=c_1\sinh\lambda x+c_2\cosh\lambda x+c_3\sin\beta x+c_4\cos\beta x$ (94)

where c_1, c_2, c_3 and c_4 are the four constants of integration.

Enforcement of Boundary Conditions

For the Kirchhoff plate considered in this study, all the four edges $x=\pm a$, $y=\pm b$ are clamped. The boundary conditions along the edges $x=\pm a$ are expressed from $u_z(x=\pm a,y)=0$ and $\frac{\partial u_z}{\partial x}(x=\pm a,y)=u_z'(x=\pm a,y)=0$ as:

$$f(x=\pm a)g(y)=0;\quad f'(x=\pm a)g(y)=0$$

Hence, since $g(y)\neq 0$, we obtain the following results:

$$f(x=\pm a)=0 \quad (95)$$

$$f'(x=\pm a)=0 \quad (96)$$

The symmetry of the considered problem suggests that $f(x)$ is a symmetric function. For $f(x)$ to be a symmetrical function in x,

$$f(x=a)=f(x=-a) \tag{97}$$

Hence,

$$f(x=a)=c_1\sinh\lambda a+c_2\cosh\lambda a+c_3\sin\beta a+c_4\cos\beta a$$
$$f(x=-a)=c_1\sinh(-\lambda a)+c_2\cosh(-\lambda a)+c_3\sin(-\beta a)+c_4\cos(-\beta a)$$
$$=c_1\sinh\lambda a+c_2\cosh\lambda a-c_3\sin\beta a+c_4\cos\beta a$$

Hence for symmetry of $f(x)$,

$$c_1\sinh\lambda a+c_2\cosh\lambda a+c_3\sin\beta a+c_4\cos\beta a$$
$$=-c_1\sinh\lambda a+c_2\cosh\lambda a-c_3\sin\beta a+c_4\cos\beta a \tag{98}$$

Simplifying,

$$c_1\sinh\lambda a+c_3\sin\beta a=-c_1\sinh\lambda a-c_3\sin\beta a \tag{99}$$

Equating the like terms yield the following two equations:

$$c_1\sinh\lambda a=-c_1\sinh\lambda a \tag{100}$$

$$c_3\sin\beta a=-c_3\sin\beta a \tag{101}$$

Solving,

$$2c_1\sinh\lambda a=0 \tag{102}$$

$$2c_3\sin\beta a=0 \tag{103}$$

Hence, $c_1=0$ (104)

Since $\sinh\lambda a\neq 0$ and

$$c_3=0 \tag{105}$$

Since $\sin\beta a\neq 0$

The same conclusion regarding the values of c_1 and c_3 is obtained if it is reasoned that the symmetry of $f(x)$ must demand that the nonsymmetrical functions in $f(x)$ must vanish. Since the sinh function and the sin function are nonsymmetrical functions in $f(x)$ their coefficients must necessarily vanish for $f(x)$ to be a symmetrical function in x.

Then $f(x)$ becomes:

$$f(x)=c_2\cosh\lambda x+c_4\cos\beta x \tag{106}$$

By differentiation of $f(x)$ with respect to x, we obtain:

$$f'(x)=c_2\lambda\sinh\lambda x-c_4\beta\sin\beta x \tag{107}$$

Determining the Eigenequation from the Conditions for Nontrivial Solutions

The boundary conditions on $f(x)$ are enforced at the clamped edges $x=\pm a$ to find the conditions for nontrivial solutions of the problem. Thus,

$$f(x=\pm a)=c_2\cosh(\pm\lambda a)+c_4\cos(\pm\beta a)$$
$$=c_2\cosh\beta a+c_4\cos\beta a=0 \tag{108}$$

$$f'(x=\pm a)=c_2\lambda\sinh(\pm\lambda a)-c_4\beta\sin(\pm\beta a)=0 \quad \ldots (109)$$

Thus,

$$f'(x=a)=c_2\lambda\sinh\lambda a-c_4\beta\sin\beta a=0 \tag{110}$$

$$f'(x=-a)=c_2\lambda\sinh(-\lambda a)-c_4\beta\sin(-\beta a)$$
$$=-c_2\lambda\sinh\lambda a+c_4\beta\sin\beta a=0 \tag{111}$$

Hence, we have the following systems of equations:

$c_2\cosh\lambda a+c_4\cos\beta a=0;\ \ c_2\lambda\sinh\lambda a-c_4\beta\sin\beta a=0$

or, $c_2\cosh\lambda a+c_4\cos\beta a=0;$

$-c_2\lambda\sinh\lambda a+c_4\beta\sin\beta a=0$

In matrix form, we write the systems of equations as:

$$\begin{pmatrix}\cosh\lambda a & \cos\beta a\\ \lambda\sinh\lambda a & -\beta\sin\beta a\end{pmatrix}\begin{pmatrix}c_2\\ c_4\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{112}$$

or, $$\begin{pmatrix}\cosh\lambda a & \cos\beta a\\ -\lambda\sinh\lambda a & \beta\sin\beta a\end{pmatrix}\begin{pmatrix}c_2\\ c_4\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{113}$$

For nontrivial solutions of Equation (112) $\begin{pmatrix}c_2\\ c_4\end{pmatrix}\neq 0$, and we obtain the eigenequation in determinant form as:

$$\begin{vmatrix}\cosh\lambda a & \cos\beta a\\ \lambda\sinh\lambda a & \beta\sin\beta a\end{vmatrix}=0 \tag{114}$$

or, from Equation (113),

$$\begin{vmatrix}\cosh\lambda a & \cos\beta a\\ -\lambda\sinh\lambda a & \beta\sin\beta a\end{vmatrix}=0 \tag{115}$$

Expansion of the determinant gives:

$$-\beta\sin\beta a\cosh\lambda a-\lambda\sinh\lambda a\cos\beta a=0 \tag{116}$$

for Equation (112), or

$$\beta\sin\beta a\cosh\lambda a+\lambda\sinh\lambda a\cos\beta a=0 \tag{117}$$

for Equation (113).

The two eigenequations - Equations (116) and (117) - are equivalent and identical. Dividing Equation (117) by $\cosh\lambda a \cos\beta a$ gives:

$$\frac{\beta \sin\beta a \cosh\lambda a + \lambda \sinh\lambda a \cos\beta a}{\cosh\lambda a \cos\beta a} = 0$$

Simplifying,

$$\frac{\beta \sin\beta a}{\cos\beta a} + \frac{\lambda \sinh\lambda a}{\cosh\lambda a} = 0$$

$$\beta \tan\beta a + \lambda \tanh\lambda a = 0$$

The obtained eigenequation for the biaxial elastic buckling problem of clamped rectangular thin plates is a transcendental equation which has no mathematical closed-form solution. The eigenequation can however be solved approximately using numerical computational algorithms for given aspect ratios of the plate.

For square plates, $a = b$, $\lambda a = 1.0046$, and $\beta a = 2.8814$.

Then,

$$\lambda = \frac{1.0046}{a}$$

$$\beta = \frac{2.8814}{a}$$

and $(\alpha a)^2 = (\alpha b)^2 = 13.2932448$

$$\alpha^2 = \frac{13.2932448}{a^2} = \frac{13.2932448}{b^2} \quad (118)$$

The critical buckling load in biaxial uniform compression is then found as:

$$\alpha^2 = \frac{N}{D} = \frac{13.2932448}{b^2} \quad (119)$$

$$N = N_{cr} = 13.2932448 \frac{D}{b^2} \quad (120)$$

$$N_{cr} = 1.346887298 \frac{D\pi^2}{b^2} \approx 1.346887 \frac{D\pi^2}{b^2} \quad (120a)$$

Other solutions for the higher buckling modes are found for square plates as

$$\alpha^2 = \frac{40}{b^2} = \frac{40}{a^2} \quad (121a)$$

$$\alpha^2 = \frac{88.70}{b^2} = \frac{88.70}{a^2} \quad (121b)$$

Reddy's approximate solution for the critical elastic buckling load using the ordinary finite difference method (FDM) for four finite difference elements for the biaxial uniform compressive load case on a square thin plate ($a \times a$) is

$$N_{cr} = 18 \frac{D}{a^2} \quad (122)$$

where the inplane dimensions are $a \times a$.

Eigenfunction

From Equation (108), we have

$$c_2 = \frac{-c_4 \cos\beta a}{\cosh\beta a} \quad (123a)$$

$$\text{or, } c_4 = \frac{-c_2 \cosh\lambda a}{\cos\beta a} \quad (123b)$$

Hence,

$$u_z = \left(c_2 \cosh\lambda x - c_2 \left(\frac{\cosh\lambda a}{\cos\beta a} \right) \cos\beta x \right) (y^2 - b^2)^2 \quad \ldots (124a)$$

$$u_z = c_2 \left(\cosh\lambda x - \frac{\cosh\lambda a}{\cos\beta a} \cos\beta x \right) (y^2 - b^2)^2 \quad \ldots (124b)$$

or,

$$u_z = c_2 \left(\frac{\cos\beta a \cosh\lambda x - \cosh\lambda a \cos\beta x}{\cos\beta a} \right) (y^2 - b^2)^2 \ldots (124c)$$

Alternatively, u_z is expressed as:

$$u_z = \left(\frac{-c_4 \cos\beta a}{\cosh\lambda a} \cosh\lambda x + c_4 \cos\beta x \right) (y^2 - b^2)^2 \quad \ldots (125a)$$

$$u_z = c_4 \left(\frac{-\cos\beta a \cosh\lambda x + \cosh\lambda a \cos\beta x}{\cosh\lambda a} \right) (y^2 - b^2)^2 \ldots (125b)$$

Solution of the Euler-Lagrange Differential Equation for Elastic Buckling under Uniaxial Uniform Compression

For equation (63) which is the Euler-Lagrange differential equation for the elastic buckling problem of clamped rectangular thin plate under uniaxial uniform compression in the x direction, the method of trial functions is used to obtain the solution. Let $f(x)$ that satisfies Equation (63) be expressible in the exponential form:

$$f(x) = \exp(\bar{s}x) = e^{\bar{s}x} \quad (126)$$

where $\bar{s}$ is a parameter we desire to find.

Then the equation reduces to the algebraic problem.

$$b^4\bar{s}^4 + (\alpha^2 b^2 - 6)b^2\bar{s}^2 + 31.5 = 0 \quad (127)$$

$$b^2\bar{s}^2 = -\left(\frac{(\alpha b)^2}{2} - 3\right) \pm \left(\left(\frac{(\alpha b)^2}{2} - 3\right)^2 - 31.5\right)^{1/2} \quad (128)$$

Solving, using the positive sign,

$$\bar{s}^2 = \left\{\frac{-\left(\frac{(\alpha b)^2}{2} - 3\right) + \left(\left(\frac{(\alpha b)^2}{2} - 3\right)^2 - 31.5\right)^{1/2}}{b^2}\right\} \quad (129)$$

Then $\bar{s} = \pm\lambda$ (130)

where,

$$\lambda = \frac{\left(\left(\frac{(\alpha b)^2}{2} - 3\right)^2 - 31.5\right)^{1/2} - \left(\frac{(\alpha b)^2}{2} - 3\right)}{b} \quad (131)$$

Using the negative sign,

$$\bar{s}^2 = \frac{-\left(\frac{(\alpha b)^2}{2} - 3\right) - \left(\left(\frac{(\alpha b)^2}{2} - 3\right)^2 - 31.5\right)^{1/2}}{b^2} \quad (132a)$$

$$\bar{s}^2 = -\beta_*^2 = -\frac{1}{b^2}\left\{\left(\frac{(\alpha b)^2}{2} - 3\right) + \left(\left(\frac{(\alpha b)^2}{2} - 3\right)^2 - 31.5\right)^{1/2}\right\} \quad (132b)$$

Hence $\bar{s} = \pm i\beta_*$ (133)

where $i = \sqrt{-1}$ (134)

i is the imaginary number, and β_* is given by:

$$\beta_* = \frac{1}{b}\left\{\left(\frac{(\alpha b)^2}{2} - 3\right) + \left(\left(\frac{(\alpha b)^2}{2} - 3\right)^2 - 31.5\right)^{1/2}\right\} \quad (135)$$

Hence the general solution for $f(x)$ in the case of uniaxial buckling is:

$$f(x) = c_1 \sinh\lambda_* x + c_2 \cosh\lambda_* x + c_3 \sin\beta_* x + c_4 \cos\beta_* x \quad (136)$$

Enforcement of Boundary Conditions at the Clamped Edges $x = \pm a$

From the symmetry of the studied problem $f(x)$ is a symmetrical function of x.
Thus,

$$f(x) = f(-x), \quad f(x = a) = f(x = -a) \quad (137)$$

Hence,

$$c_1 \sinh\lambda_* x + c_2 \cosh\lambda_* x + c_3 \sin\beta_* x + c_4 \cos\beta_* x = c_1 \sinh(-\lambda_* x) + c_2 \cosh(-\lambda_* x) + c_3 \sin(-\beta_* x) + c_4 \cos(-\beta_* x) \quad (138)$$

$$c_1 \sinh\lambda_* x + c_2 \cosh\lambda_* x + c_3 \sin\beta_* x + c_4 \cos\beta_* x = -c_1 \sinh\lambda_* x + c_2 \cosh\lambda_* x - c_3 \sin\beta_* x + c_4 \cos\beta_* x \quad (139)$$

Simplifying,

$$c_1 \sinh\lambda_* x + c_3 \sin\beta_* x = -c_1 \sinh\lambda_* x - c_3 \sin\beta_* x \quad (140)$$

Equating like times,

$$c_1 \sinh\lambda_* x = -c_1 \sinh\lambda_* x \quad (141a)$$

$$c_3 \sin\beta_* x = -c_3 \sin\beta_* x \quad (141b)$$

Solving, $2c_1 \sinh\lambda_* x = 0$

$$\therefore c_1 = 0 \quad (142a)$$

$$2c_3 \sin\beta_* x = 0$$
$$c_3 = 0 \quad (142b)$$

Then,

$$f(x) = c_2 \cosh\lambda_* x + c_4 \cos\beta_* x \quad (143)$$

Differentiating $f(x)$ with respect to x, we obtain:

$$f'(x) = c_2\lambda_* \sinh\lambda_* x - c_4\beta_* \sin\beta_* x \quad (144)$$

Imposition of the boundary conditions at the clamped edges $x = \pm a$ gives the following system of two equations:

$$f(x = \pm a) = c_2 \cosh(\pm\lambda_* a) + c_4 \cos(\pm\beta_* a) = 0 \quad (145)$$
$$f'(x = \pm a) = c_2\lambda_* \sinh(\pm\lambda_* a) - c_4\beta_* \sin(\pm\beta_* a) = 0 \quad (146)$$

Thus,

$$c_2 \cosh\lambda_* a + c_4 \cos\beta_* a = 0 \quad (147a)$$

$$c_2\lambda_* \sinh\lambda_* a - c_4\beta_* \sin\beta_* a = 0 \quad (147b)$$

or,

$$c_2 \cosh\lambda_* a + c_4 \cos\beta_* a = 0 \quad (147a)$$

$$-c_2\lambda_* \sinh\lambda_* a + c_4\beta_* \sin\beta_* a = 0 \quad (147c)$$

In matrix form,

$$\begin{pmatrix} \cosh\lambda_* a & \cos\beta_* a \\ \lambda_* \sinh\lambda_* a & -\beta_* \sin\beta_* a \end{pmatrix}\begin{pmatrix} c_2 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (148a)$$

or, $$\begin{pmatrix} \cosh\lambda_* a & \cos\beta_* a \\ -\lambda_* \sinh\lambda_* a & \beta_* \sin\beta_* a \end{pmatrix}\begin{pmatrix} c_2 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (148b)$$

For nontrivial solutions of the system of homogeneous equations, $\begin{pmatrix} c_2 \\ c_4 \end{pmatrix} \neq 0$ and we obtain the following determinantal equation as the eigenequation:

$$\begin{vmatrix} \cosh\lambda_* a & \cos\beta_* a \\ \lambda_* \sinh\lambda_* a & -\beta_* \sin\beta_* a \end{vmatrix} = 0 \quad (149a)$$

or, $$\begin{vmatrix} \cosh\lambda_* a & \cos\beta_* a \\ -\lambda_* \sinh\lambda_* a & \beta_* \sin\beta_* a \end{vmatrix} = 0 \quad (149b)$$

Expanding the determinant gives the characteristic stability equation as follows:

$$-\beta_* \sin\beta_* a \cosh\lambda_* a - \lambda_* \sinh\lambda_* a \cos\beta_* a = 0 \quad \ldots (150a)$$

or, $$\beta_* \sin\beta_* a \cosh\lambda_* a + \lambda_* \sinh\lambda_* a \cos\beta_* a = 0 \quad \ldots (150b)$$

Both eigenequations-Equations (150a) and (150b) - are identical. Dividing Equation (150b) by $\cosh\lambda_* a \cos\beta_* a$ gives:

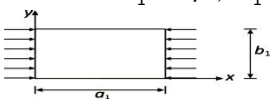
$$\frac{\beta_* \sin\beta_* a \cosh\lambda_* a + \lambda_* \sinh\lambda_* a \cos\beta_* a}{\cosh\lambda_* a \cos\beta_* a} = 0 \quad (151a)$$

$$\beta_* \tan\beta_* a + \lambda_* \tanh\lambda_* a = 0 \quad (151b)$$

The characteristic buckling equation thus obtained is a transcendental equation with infinitely many solutions called the roots, zeros or eigenvalues of the equation. The transcendental equation is solved using iteration based computer software applications to find the roots for any corresponding buckling modes.

From the eigenvalues the critical buckling loads are calculated and presented in tables for various values of the plate aspect ratio (*a*/*b*). Table-1 shows the elastic buckling load coefficient (factors) obtained in the present study for various values of the plate aspect ratio. Table-1 also presents the elastic buckling load coefficients obtained by previous researchers who used various other research methods for the considered problem of buckling of clamped plates under uniaxial uniform compression.

Table-1. Elastic buckling coefficients (factors) $K(a_1/b_1)$ for clamped rectangular thin plate under uniaxial uniform compressive load $a_1 = a/2,\ b_1 = b/2$.



(a_1/b_1) aspect ratio	Number of buckles	Present study $K(a_1/b_1)$	Levy [28, 29] *K*	Maulbetsch cited in Jang [2] *K*	Faxem cited in Jang [2] *K*	Iyengar [6] *K*	Ibearugbulem [8] *K*
0.50	1	18.30			18.30	19.667	20.384
0.75	1	11.66	11.66	12.77	11.39		
1.00	1	10.07	10.07	10.48	10.07	10.667	10.878
1.25	2	9.25	9.25	9.38	9.20		
1.50	2	8.33	8.33	8.45	8.30		
1.75	2	8.11	8.11	8.17	8.18		
2	3	7.88	7.88	8.06	7.87		
2.25	3	7.63	7.63	7.96			
2.50	3	7.57	7.57	7.99			
2.75	4	7.44	7.44	7.76			
3	4	7.37	7.37	7.59			
3.25	4	7.35	7.35	7.86			
3.50	5	7.27	7.27	7.37			
3.75	5	7.24	7.24	7.40			
4	5	7.23	7.23	7.45			

Eigenfunction for Elastic Buckling of Clamped Rectangular Thin Plate under Uniaxial Uniform Compressive Load

From the boundary condition equation, Equation (147a), we have:

$$c_2 = \frac{-c_4 \cos\beta_* a}{\cosh\lambda_* a} \tag{152a}$$

$$\text{or } c_4 = \frac{-c_2 \cosh\lambda_* a}{\cos\beta_* a} \tag{152b}$$

Then,

$$u_z(x,y) = \left(c_2 \cosh\lambda_* x - \frac{c_2 \cosh\lambda_* a}{\cos\beta_* a}\cos\beta_* x\right)(y^2 - b^2)^2 \tag{153a}$$

The eigenfunction becomes:

$$u_z(x,y) = c_2\left(\cosh\lambda_* x - \frac{\cosh\lambda_* a}{\cos\beta_* a}\cos\beta_* x\right)(y^2 - b^2)^2 \tag{153b}$$

Alternatively the eigenfunction can be expressed as:

$$u_z(x,y) = c_4\left(\frac{-\cos\beta_* a}{\cosh\lambda_* a}\cosh\lambda_* x + \cos\beta_* x\right)(y^2 - b^2)^2 \tag{154a}$$

$$u_z = c_4\left(\frac{-\cos\beta_* a \cosh\lambda_* x + \cos\beta_* x \cosh\lambda_* a}{\cosh\lambda_* a}\right)(y^2 - b^2)^2 \tag{154b}$$

DISCUSSIONS

This work has presented in a systematic way, the variational Ritz-Kantorovich-Euler Lagrange method for the elastic buckling of clamped rectangular thin plate. Two cases of buckling considered were due to biaxial uniform compressive loads in the x and y coordinate directions of the plate and uniaxial uniform compressive load in the x coordinate direction of the plate.

The theoretical framework of the study is based on energy minimization principles, wherein the total potential energy functionals for the rectangular thin plate buckling problem for the two considered cases of biaxial and uniaxial buckling were to be minimized.

The problems solved in this work are (i) the elastic buckling problem of rectangular thin plate with clamped edges $x = \pm a$, $y = \pm b$ as shown in Figure-1 with the edges $x = \pm a$ subject to compressive load N_x and the edges $y = \pm b$ subjected to compressive load N_y where for this work $N_x = N_y = N$ and (ii) the elastic buckling problem of rectangular thin plate with clamped edges $x = \pm a$ as shown in Figure 2 where the edges $x = \pm a$ are subjected to compressive load N_x.

For the case of biaxial buckling, the total potential energy functional Π for homogeneous, isotropic plates is found as the sum of the strain energy and the potential energy of loads as Equation (8).

Following Kantorovich's approach the unknown deflection function $u_z(x,y)$ is chosen in the coordinate variables-separable form as the product of two functions $f(x)$ and $g(y)$ where one function depends only on x and the other function depends only on y.

The coordinate basis function in the y direction $g(y)$ is derived from a polynomial function of fourth degree using the demands of symmetry of $g(y)$ and the boundary conditions along the clamped edges $y = \pm b$. The coordinate basis function derived is given by Equation (31), and thus satisfies all the boundary conditions at $y = \pm b$ and the requirements of symmetry.

The deflection is thus found as Equation (32) which contains an unknown function $f(x)$. Equation (32) is used to derive the modified Ritz functional Π_* as Equation (37). Equation (37) is expressed in general as the variational problem of minimizing a functional with respect to an unknown function $f(x)$ as Equation (42) where the integrand $F(x, f(x), f'(x), f''(x))$ is given by Equation (43).

The condition for the minimization of the modified Ritz functional is the Euler-Lagrange differential equation given by Equation (44). The Euler-Lagrange equation was found explicitly for the biaxial buckling problem as the fourth order ODE given by Equation (55). The method of trial functions is used to solve Equation (55) by considering the trial solution in exponential form.

From conditions of nontrivial solutions, the ODE is reduced to an algebraic problem given by Equation (66). Quadratic formula was used to obtain the four roots as Equations (78) and (85). The general solution for $f(x)$ is then found as Equation (94). The requirements of symmetry of $f(x)$ was used to obtain two integration constants as Equations (104) and (105) and $f(x)$ as Equation (106).

The boundary conditions at the clamped edges $x = \pm a$ were used to set-up a system of two homogeneous equations given in matrix form as Equation (112) or (113). The requirement of nontrivial solutions of the matrix equation was used to obtain the eigenequation in determinant form as Equation (114). Expansion of the determinant and simplification gave the elastic buckling equation as the transcendental equation given by Equation (117) or (117a). The transcendental equation has no closed-form solution but can be solved approximately using numerical computational algorithms. The equation was solved for square thin plates (for which the aspect ratio is one) and α^2 computed as Equation (118). The critical buckling load was thus obtained for square thin plate under biaxial uniform compressive load as Equation (120). The eigenfunctions for clamped rectangular thin plate under biaxial buckling was obtained by solving the eigenvalue eigenvector problem as Equation (124a) or (125b).

Discussion on Uniaxial Buckling

In a similar way, the total potential energy functional for uniaxial buckling is given by Equation (56) from the sum of the strain energy functional and the potential of the load. The variable separable form of the deflection is substituted into the total potential energy functional and after simplification; the modified Ritz functional was obtained as Equation (59) which is presented as a classic problem of functional extremization in Equation (60) where the integrand is expressed as Equation (61).

The Euler-Lagrange differential equation which is the condition for the extremum (minimum) of the modified Ritz functional is found as Equation (62) and Equation (63) after simplification. The method of trial functions is used to solve Equation (63) by assuming a trial solution in exponential form. This reduced the ODE - Equation (63) - to the algebraic problem given by Equation (127). The solution by quadratic formula gave the four roots as Equations (130) and (133). The general solution for $f(x)$ was thus obtained for uniaxial buckling as Equation (136). The requirement of symmetry was used to obtain two integration constants c_1 and c_3 as Equations (142a) and (142b) respectively. Then $f(x)$ was found as Equation (143) which has two unknown integration constants. Imposition of boundary conditions at the clamped edges $x = \pm a$ led to the system of homogeneous equations - Equations (145) and (146) or Equations (147a), (147b) or Equations (147a), (147c).

The boundary conditions are expressed in matrix form as Equations (148a) or (148b). The conditions of nontrivial solutions of the boundary conditions equations led to the eigenequation written in determinant form as Equations (149a) or (149b). Expanding the determinant and simplifying gave the eigenequation as Equation (150b) or (151b). The eigenfunction was also found by solving the eigenvalue eigenvector problem as Equations (153b) or (154b). The eigenequation is a transcendental equation with infinitely many zeros. The eigenequation is solved using numerical computational algorithms to obtain the eigenvalues from which the buckling loads are determined for various values of the plate aspect ratios.

The solutions for the elastic buckling load coefficients obtained in the present work and results from similar studies by Levy, Maulbetsch, Faxem, Iyengar and Ibearugbulem are presented in Table-1. Table-1 illustrates that the present results are identical with previous results presented by Levy (1942a, 1942b); and vary minimally with results presented by Faxem, as cited in Jang (2005).

CONCLUSIONS

a) The variational Ritz-Kantorovich-Euler-Lagrange method has been successfully used in this article to obtain exact solutions for the eigenequations and eigenfunctions of fully clamped rectangular thin plate for the two cases where the plate is under biaxial uniform compression and under uniaxial uniform compression.

b) Implementation of the Kantorovich approach in the construction of the unknown deflection as the product of one unknown function $f(x)$ and a coordinate (basis) function in the y direction $g(y)$ simplified the total potential energy functional for the elastic stability problem to depend only on $f(x)$ and derivatives of $f(x)$.

c) The modified functional expression is minimized for each considered case by using the Euler-Lagrange differential equation, which is the sufficient and necessary condition for the extremum of the total potential energy functional.

d) The Euler-Lagrange differential equation obtained for each considered case is a fourth order linear differential equation which could be solved in closed form for $f(x)$ using such classical methods for solving ODEs as trial function method, variation of parameters, and differential (D) operator methods.

e) The use of trial function methods that assume the solution in exponential form reduced the ODE problem to algebraic equations upon invoking the condition for nontrivial solutions.

f) The roots, (zero) of the resulting algebraic equation were used to obtain the basis of linearly independent solutions to the problem.

g) The solution of the ODEs for each considered case studied in the paper gave the expression for $f(x)$ in terms of hyperbolic and trigonometric functions with four constants of integration corresponding to the fourth order of the ODE.

h) The requirements of symmetry of the function $f(x)$ following from the symmetry of the problem was used to determine two constants of integration in the expression for $f(x)$ in each considered buckling problem.

i) The imposition on $f(x)$ of boundary conditions following from the vanishing of the displacements and rotations of the clamped edges ($x = \pm a$) was used to obtain a system of equations for the boundary conditions at the clamped edges ($x = \pm a$).

j) The conditions for nontrivial solution of the system of equations was applied to the system of boundary conditions equations at $x = \pm a$ to obtain the eigenequation, upon expansion of the determinant.

k) The eigenequation is a transcendental equations which is solved for each case of biaxial and uniaxial uniform

compression to obtain the eigenvalues from which the buckling loads are computed for assumed aspect ratios of the plate.

l) The eigenfunction is determined from the system of boundary conditions equations and thus by solving the eigenvalue - eigenvector problem.

m) The resulting solution for the critical elastic buckling load for fully clamped square thin plate under biaxial uniform compressive load is identical with results obtained by previous researchers, thus validating the work.

n) The present solutions for the critical elastic buckling loads of rectangular thin plate with fully clamped edges are in excellent agreement with results obtained by Levy (1942a, 1942b), thus validating the work.

o) The study presents exact or mathematical closed-form solutions to the problem because (a) $g(y)$ satisfies all the boundary conditions along the clamped edges ($y = \pm b$); (b) $f(x)$ is found such that the functional Π_* is minimized at every point in the plate domain and this corresponds to the Euler-Lagrange differential equation being satisfied identically at every point in the solution domain; (c) $f(x)$ satisfies all the boundary conditions along the clamped edges ($x = \pm a$).

Nomenclature/Notations

x, y	inplane Cartesian coordinates of the plate.
$u_z(x, y)$	deflection which is a function of the x and y coordinates on the plate domain.
D	modulus of flexural rigidity of the plate.
N_x	inplane edge compressive force in the x coordinate direction.
N_y	inplane edge compressive force in the y coordinate direction.
N_{xy}	inplane edge shearing (shear) force.
E	Young's modulus of elasticity of the material of the plate.
t	thickness of the plate
μ	Poisson's ratio of the material of the plate.
a	half of the inplane dimension in the x direction of the plate
b	half of the inplane dimension in the y direction of the plate.
Π	total potential energy functional for rectangular thin plate in elastic stability problem.
U	strain energy functional for plate bending.
V	potential energy functional due to applied edge loads acting on the plate.
N	uniformly distributed biaxial compressive load applied in the x and y coordinate directions.
$g(y)$	coordinate (basis) function in the y direction for the plate deflection.
$f(x)$	basis function in the x coordinate direction for the plate deflection.
e_1, e_2, e_3, e_4, e_5	polynomial constants in the fourth degree polynomial used in determining the basis function in x, y direction for plate deflection.
$g_1(y)$	fourth degree polynomial used in determining $g(y)$.
Π_*	modified Ritz functional for the elastic buckling problem of rectangular thin plate assumed to be homogenous and isotropic,
α^2	parameter defined in terms of N and D.
I_1, I_2, I_3, I_4	definite integrals defined over $-b \le y \le b$ in terms of $g(y)$, $g''(y)$, and $g'(y)$.
$F(x, f(x), f'(x), f''(x))$	integrand in the functional Π^* (defined over $-a \le x \le a$).
$\exp(sx)$	exponential function of sx
s	undetermined and unknown parameter of the trial function $f(x)$ which is sought for determination.
N_{cr}	critical buckling load
$i = \sqrt{-1}$	imaginary number or complex number
$\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$	integration constants when $f(x)$ is expressed as exponential functions.
c_1, c_2, c_3, c_4	integration constants when $f(x)$ is expressed in terms of trigonometric and hyperbolic functions.
$\pm\lambda, \pm i\beta$	roots of the fourth order algebraic polynomial (characteristic buckling equation) for biaxial buckling.
$\pm\lambda_*, \pm i\beta_*$	roots of the characteristic auxiliary polynomial for uniaxial buckling problem considered.
Subscripts: cr	critical.
ODE(s)	ordinary differential equation(s)
FDM	finite difference method
Sin	sin function
Cos	cosine function
Sinh	hyperbolic sine function
Cosh	hyperbolic cosine function
∇^2	Laplacian operator
∇^4	biharmonic operator
$\int_{-a}^{a} (\ )\, dx$	integral defined over $-a \le x \le a$ or integration with respect to x between the limits of $x = -a$ to $x = a$.
$f'(x) = \dfrac{df(x)}{dx}$	first derivative of $f(x)$ with respect to x.
$f^n(x) = \dfrac{d^n f(x)}{dx^n}$	nth derivative of $f(x)$ with respect to x.
$\lvert\ \rvert$	determinant.
$\dfrac{\partial}{\partial x}$	partial derivative with respect to x.

$\frac{\partial^n}{\partial x^n}$	nth partial derivative with respect to x.
$\pm$	plus or minus
$\int_{-a}^{a}\int_{-b}^{b}(\ )dxdy$	double integration over the domain $-a \le x \le a;\ -b \le y \le b.$
$\sqrt{\ }$	square root

Conflict of interest

The author declares no conflict of interests.

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