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SUMUDU TRANSFORM METHOD FOR FINDING THE TRANSVERSE NATURAL HARMONIC VIBRATION FREQUENCIES OF EULER-BERNOULLI BEAMS

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ABSTRACT

The determination of the natural frequencies of flexural vibrations of Euler-Bernoulli beams is a vital consideration in their analysis and design for dynamic loads. This paper presents the Sumudu transform method for the determination of the natural frequencies of Euler-Bernoulli beams under transverse free harmonic vibration for different boundary conditions. The end support conditions considered are: (a) simply supported at both bends, (b) clamped at both ends, (c) clamped-free ends (d) clamped-simply supported ends, and (e) simply supported-clamped ends. The governing partial differential equation is converted by the Sumudu transformation to an integral equation, which upon evaluation becomes an algebraic equation. The solution gives the dynamic modal displacement shape function in the Sumudu transform space $V(u)$. Inversion gives the dynamic modal displacement function in the physical problem space $V(x)$. The enforcement of boundary conditions for the end supports considered yielded systems of homogeneous equations. The condition for nontrivial solutions is used to determine the characteristic frequency equation for each considered boundary condition. It is found that the characteristic frequency equation has an infinite number of eigenvalues (roots or zeros) corresponding to the continuously distributed parameter model of idealization of the problem. The characteristic frequency equations obtained are solved for the n roots using computational software methods, Symbolic Algebra Software and Mathematica Software to obtain the eigenvalues (zeros or roots) for any (n) vibration mode. The eigenvalues are then used to obtain the eigenfrequencies or natural frequencies of flexural vibration for each considered boundary conditions. It is found that closed form solutions obtained are identical to the solutions in the literature; obtained by classical methods of separation of variables and eigenfunction expansion methods.

Keywords: characteristic frequency equation, eigenvalue, Euler-Bernoulli beam theory, flexural vibration mode, natural frequencies, Sumudu transform method.

1. INTRODUCTION

The transverse (or flexural) free vibration problems of beams have important applications in many fields of engineering including civil engineering, structural engineering and mechanical engineering. The governing equations of flexural vibrations of beams depend on the beam theory used. Beam theories that are frequently used are: (i) Euler-Bernoulli beam theory (EBT), (ii) Timoshenko beam theory, (iii) Mindlin beam theory, (iv) shear deformation beam theories.

Euler-Bernoulli beam theory, which is adopted in this work disregards the effects of transverse shear deformation and rotary inertia in the formulation of the governing differential equations [1-10], and hence it's application only to thin (slender) beams. The theory assumes that plane cross sections remain plane and perpendicular to the neutral axis during flexural deformation.

Timoshenko beam theory, Mindlin beam theory and other shear deformation beam theories take cognizance of the shear deformation and rotary inertia in the development of the governing equations, and hence can be used to model and describe moderately thick and thick beams [1-6]. Timoshenko beam theory retains the assumption that the cross-section remains plane during flexural deformation. However, the assumption of orthogonality of the cross-section to the neutral axis is relaxed in the Timoshenko theory.

Methods used in the literature for the formulation of the governing equations of beam vibration are: the equilibrium methods and variational methods. In the equilibrium methods, D'Alembert's principle of dynamic equilibrium is used to obtain the governing equation as a differential equation. In the variational methods, the calculus of variations is used to express the problem in terms of functionals which are integrals to be minimized for equilibrium. Variational techniques use the energy minimization principles to formulate the governing equations as integral statements.

The governing equations for transverse vibrations of beams are solved using methods for solving differential equations when expressed as differential equations, or methods for solving integral equations when expressed as variational statements. Hence the following solution methods have been used for solving the Euler-Bernoulli beam flexural vibration problems:

a) Separation of variables (product) method

b) Eigenfunction expansion methods

c) Laplace Transform Method (LTM)

d) Differential Transform Method (DTM)

e) Adomian Decomposition Method (ADM)

f) Variational Iteration Method (VIM)

g) Homotopy Perturbation Method (HPM)

h) Point Interpolation Method (PIM)

i) Radial Point Interpolation Method (RPIM)

j) Finite Element Method (FEM)

k) Finite Difference method (FDM)

l) Improved Finite Difference Method (IFDM)

Presently, to the author's knowledge, the Sumudu Transform Method has not been previously used to solve the transverse free vibration problem of Euler-Bernoulli beams.

Chun [11] investigated the natural vibrations of an Euler-Bernoulli beam with one end free and the other end hinged by a rotational spring with constant spring stiffness. Agboola and Gbadeyan [12] used the differential transformation method to solve the free vibration problem of Euler-Bernoulli beam under various end support conditions and obtained solutions that are consistent with the solutions obtained using the other methods from literature. Kanbar and Tufik [13] analysed the vibration of thin beams by using Point Interpolation Method (PIM) and Radial Point Interpolation Method (RPIM) with Standard Gaussian integration and a nodal integration based on the Taylor series expansion. They investigated the effects of integration schemes, support, domain sizes and RPIM shape parameters on the vibration on the vibration modes. They solved the prismatic cantilever vibration problem and found agreement with previously obtained finite element and analytical solutions.

Wang and Lin [14] applied the Fourier series technique to solve the dynamic problem of beams under various boundary conditions. Kim and Kim [15] also applied the Fourier series method to determine the eigenfrequencies of uniform beams under various restraint conditions. Lai *et al* [16] solved the free vibration problem of homogeneous prismatic Euler-Bernoulli beam with different elastically supported conditions using Adomian decomposition method. Li [17] studied the vibration characteristics of a beam having general boundary conditions. Achawakorn and Jearsiripongkul [18] used the Galerkin's method to study the free vibration problems of uniform and non-uniform beam (with exponential cross section). They determined the natural frequencies that were comparable to previous results obtained using finite element method (FEM). Lin and Gurram [19] used the He's variational iteration method (VIM) to determine the eigenfrequencies and mode shapes of an Euler-Bernoulli beam under various end restraint conditions. Coskun *et al* [20] used Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) to solve the free transverse flexural vibration problem of an Euler-Bernoulli beam for various end support conditions. They presented illustrative case studies to validate their work by comparison with exact solutions, and obtained close agreement with exact solutions. Agboola *et al* [21] used the variational iteration method (VIM) to determine the natural frequencies of a cantilever beam with a prismatic cross-section, undergoing flexural (transverse) vibrations. They validated their results by comparison with previous results obtained by closed form solutions of the governing equations of motion, and differential transform method (DTM). Sakman and Mutlu [22] used a perturbation technique to determine the natural frequencies and mode shapes of Euler-Bernoulli beam with a rectangular cross-section which has a surface crack assumed much smaller than the beam cross-section.

Hurty and Rubinstein [23] studied the dynamics of structures. Simsek and Kocatürk [24] used a third order shear deformation theory to study the natural vibrations of beams. Mukherjee and Gosai [25] determined the natural frequencies of Euler-Bernoulli beams using analytical and finite elements method, and presented solutions for various end support conditions.

2. JUSTIFICATION FOR ADOPTING THE SUMUDU TRANSFORM METHOD (STM)

The Sumudu Transform method is adopted in this work because it has the following advantages over the classical methods for solving Boundary Value Problems (BVPs):

a) The STM gives a complete solution to the governing equation.

b) Initial conditions are automatically considered in the Sumudu transformed domain equations.

c) Much less time and computational efforts is involved in solving the differential equations governing the problem since they are transformed to algebraic polynomial equations that are relatively easier to solve.

d) The STM presents systematic and routine solutions to the initial value problem (IVP).

The disadvantage of the STM is that in some cases, the inversion of the transform is difficult to obtain, since it involves complex problems of contour integration. However, the relationship between the STM and the Laplace transform method suggests that some difficult inversion problems could be solved using catalogues of known Laplace transform inverses.

In this paper, the Sumudu transform method is used to solve the free vibration problem of Euler-Bernoulli beams for the boundary conditions of:

a) Simply supported ends ($x = 0$, and $x = l$)

b) Clamped ends (x = 0, and x = l)

c) Cantilever Euler-Bernoulli beam

d) End x = 0 is clamped and end x = l is simply supported, and

e) End x = 0 is simply supported and end x = l is clamped.

3. METHODOLOGY

The governing partial differential equation (PDE) for the natural vibration of Euler-Bernoulli beam is:

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + m\frac{\partial^2 v(x,t)}{\partial t^2} = 0 \tag{1}$$

where $v(x, t)$ is the transverse displacement which depends upon x, and t. x is the longitudinal (axial) coordinate of the beam, t is the time, m is the mass per unit length of the beam, E is the Young's modulus of elasticity, I is the moment of inertia of the beam cross-section.

For harmonic vibrations, $v(x, t)$ is expected to be a harmonic function of time.

Let $v(x,t) = V(x)\sin(\omega_n t - \phi)$ (2)

where $V(x)$ is the dynamic displacement modal shape function, ω_n is the natural frequency, ϕ is the phase.

Then the governing PDE is:

$$\left(EI\,V^{iv}(x) - m\omega_n^2 V(x)\right)\sin(\omega_n t - \phi) = 0 \tag{3}$$

Then,

$$V^{iv}(x) - \frac{m\omega_n^2}{EI}V(x) = 0 \tag{4}$$

$$V^{iv}(x) - \alpha_n^4 V(x) = 0 \tag{5}$$

where $\alpha_n^4 = \frac{m\omega_n^2}{EI}$ (6)

This paper solves Equation (5) by the Sumudu transform method presented for solving differential equations [26-31]. Taking the Sumudu transform of Equation (5),

$$\frac{1}{u}\int_0^\infty \left(V^{iv}(x) - \alpha^4 V(x)\right)e^{-x/u}dx = 0 \tag{7}$$

where u is the Sumudu transform parameter.

Evaluation of the integral equation -Equation (7) gives:

$$\frac{V(u)}{u^4} - \frac{V(0)}{u^4} - \frac{V'(0)}{u^3} - \frac{V''(0)}{u^2} - \frac{V'''(0)}{u} - \alpha^4 V(u) = 0 \tag{8}$$

where $V(u) = \frac{1}{u}\int_0^\infty V(x)e^{-x/u}dx = SV(x)$ (9)

$V(u)$ is the Sumudu transform of $V(x)$, S is the Sumudu transform operator.

Solving for $V(u)$,

$$V(u) = \frac{u^4}{1-\alpha^4u^4}\left(\frac{V(0)}{u^4} + \frac{V'(0)}{u^3} + \frac{V''(0)}{u^2} + \frac{V'''(0)}{u}\right) \ldots(10)$$

$$V(u) = \frac{V(0)}{1-\alpha^4u^4} + \frac{uV'(0)}{1-\alpha^4u^4} + \frac{u^2V''(0)}{1-\alpha^4u^4} + \frac{u^3V'''(0)}{1-\alpha^4u^4} \ldots(11)$$

By inversion of $V(u)$, $V(x)$ is obtained.

$$V(x) = S^{-1}V(u) = V(0)S^{-1}\left(\frac{1}{1-\alpha^4u^4}\right) + V'(0)S^{-1}\frac{u}{1-\alpha^4u^4} + V''(0)S^{-1}\frac{u^2}{1-\alpha^4u^4} + V'''(0)S^{-1}\frac{u^3}{1-\alpha^4u^4} \tag{12}$$

Hence,

$$V(x) = V(0)\left(\frac{\cosh\alpha x + \cos\alpha x}{2}\right) + V'(0)\left(\frac{\sinh\alpha x + \sin\alpha x}{2\alpha}\right) + V''(0)\left(\frac{\cosh\alpha x - \cos\alpha x}{2\alpha^2}\right) + V'''(0)\left(\frac{\sinh\alpha x - \sin\alpha x}{2\alpha^3}\right) \tag{13}$$

4. RESULTS

A. Euler-Bernoulli beams with simply supported ends x = 0, x = l

Figure-1. Simply supported Euler-Bernoulli beam.

The boundary conditions of an Euler-Bernoulli beam simply supported at x = 0, and x = l as shown in Figure-1 are:

$$v(0) = v(l) = 0 \tag{14}$$

$$v''(0) = v''(l) = 0 \tag{15}$$

The boundary conditions yield the system of homogeneous algebraic equations:

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$$\begin{pmatrix} \left(\frac{\sinh \alpha l + \sin \alpha l}{2\alpha}\right) & \left(\frac{\sinh \alpha l - \sin \alpha l}{2\alpha^3}\right) \\ \alpha\left(\frac{\sinh \alpha l - \sin \alpha l}{2}\right) & \left(\frac{\sinh \alpha l + \sin \alpha l}{2\alpha}\right) \end{pmatrix} \begin{pmatrix} V'(0) \\ V'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ldots(16)$$

For nontrivial solutions,

$$\begin{pmatrix} V'(0) \\ V'''(0) \end{pmatrix} \neq 0$$

The characteristic frequency equation becomes:

$$\begin{vmatrix} \left(\frac{\sinh \alpha l + \sin \alpha l}{2\alpha}\right) & \left(\frac{\sinh \alpha l - \sin \alpha l}{2\alpha^3}\right) \\ \alpha\left(\frac{\sinh \alpha l - \sin \alpha l}{2}\right) & \left(\frac{\sinh \alpha l + \sin \alpha l}{2\alpha}\right) \end{vmatrix} = 0 \quad (17)$$

Expansion yields the solvable transcendental equation:

$$\sin \alpha l = 0 \quad (18)$$

Then, closed form solutions to the characteristic frequency equation are:

$$\alpha l = \sin^{-1}(0) = n\pi \; (n = 1, 2, 3, 4, 5, \ldots) \quad (19)$$

$$\alpha = \frac{n\pi}{l} = \left(\frac{m\omega_n^2}{EI}\right)^{1/4} \quad (20)$$

Solving for ω_n,

$$\omega_n = \left(\frac{n\pi}{l}\right)^2 \sqrt{\frac{EI}{m}} \quad (21)$$

The natural frequencies of simply supported Euler-Bernoulli beams are presented in Table-1 for the first four vibration modes.

Table-1. Natural frequencies of Euler-Bernoulli beam simply supported at x = 0 and x = l.

Vibration mode, *n*	**Avcar [9]; Hurty and Rubinstein [23]**	**Natural frequency** $\left(\times\sqrt{\frac{EI}{ml^4}}\right)$ ω_n
1	9.869604401	9.869604401
2	39.4784176	39.4784176
3	88.82643961	88.82643961
4	157.9136704	157.9136704

B. Euler-Bernoulli beams with fixed ends x = 0, x = l

The Euler-Bernoulli beam fixed at $x = 0$ and $x = l$ as shown in Figure-2 is considered.

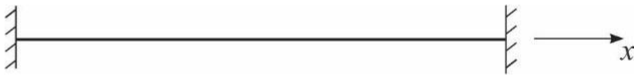

Figure-2. Euler-Bernoulli beam with fixed ends.

The boundary conditions are:

$$V(0) = V'(0) = 0 \quad (22)$$

$$V(l) = V'(l) = 0 \quad (23)$$

From the boundary conditions, the system of homogeneous equations is obtained:

$$\begin{pmatrix} \frac{\cosh \alpha x - \cos \alpha x}{2\alpha^2} & \frac{\sinh \alpha x - \sin \alpha x}{2\alpha^3} \\ \frac{\sinh \alpha x + \sin \alpha x}{2\alpha} & \frac{\cosh \alpha x - \cos \alpha x}{2\alpha^2} \end{pmatrix} \begin{pmatrix} V''(0) \\ V'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ldots (24)$$

For nontrivial solutions, the characteristic frequency equation is found as:

$$\begin{vmatrix} \frac{\cosh \alpha x - \cos \alpha x}{2\alpha^2} & \frac{\sinh \alpha x - \sin \alpha x}{2\alpha^3} \\ \frac{\sinh \alpha x + \sin \alpha x}{2\alpha} & \frac{\cosh \alpha x - \cos \alpha x}{2\alpha^2} \end{vmatrix} = 0 \quad (25)$$

Expansion gives identical expression obtained by Avcar [9]:

$$1 - \cosh \alpha x \cos \alpha x = 0 \quad (26)$$

The eigenvalues of the transcendental equation are obtained using Symbolic Algebra Software, Mathematica Software or other numerical analysis tools as:

$$\left.\begin{aligned} \alpha_1 l &= 4.73014 \\ \alpha_z l &= 7.85321 \\ \alpha_3 l &= 10.9956 \\ \alpha_4 l &= 14.1372 \\ \alpha_n l &= \frac{(2n+1)\pi}{2} \end{aligned}\right\} \quad (27)$$

The natural frequency for the nth mode is

$$\omega_n = \alpha_n^2 \sqrt{\frac{EI}{m}} \quad (28)$$

The natural frequencies of clamped-clamped Euler-Bernoulli beams are presented in Table-2 for the first four vibration modes.

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Table-2. Natural frequencies of clamped-clamped beams for the first four vibration modes.

Vibration mode, n	Avcar [9]; Hurty and Rubinstein [23]	Natural frequency $\left(\times\sqrt{\frac{EI}{ml^4}}\right)\omega_n$
1	22.37384601	22.37384601
2	61.67275024	61.67275024
3	120.9032194	120.9032194
4	199.8604238	199.8604238

C. Euler-Bernoulli beam with clamped-free ends

The Euler-Bernoulli beam clamped at x = 0 and free at x = l as shown in Figure-3 is considered.

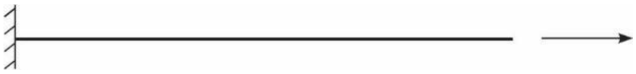

Figure-3. Euler-Bernoulli beam with clamped-free ends.

The boundary conditions are:

$$V(0) = 0 \tag{29}$$

$$\theta(0) = V'(0) = 0 \tag{30}$$

$$M(x = l) = 0 \tag{31}$$

$$V''(l) = 0 \tag{32}$$

$$Q(x = l) = 0 \tag{33}$$

$$V'''(l) = 0 \tag{34}$$

The system of homogeneous equations is:

$$\begin{pmatrix} \frac{\cosh\alpha l + \cos\alpha l}{2} & \frac{\sinh\alpha l + \sin\alpha l}{2\alpha} \\ \alpha\left(\frac{\sinh\alpha l - \sin\alpha l}{2}\right) & \frac{\cosh\alpha l + \cos\alpha l}{2} \end{pmatrix}\begin{pmatrix} V''(0) \\ V'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ldots(35)$$

For nontrivial solutions, the characteristic frequency equation is:

$$\begin{vmatrix} \frac{\cosh\alpha l + \cos\alpha l}{2} & \left(\frac{\sinh\alpha l + \sin\alpha l}{2\alpha}\right) \\ \alpha\left(\frac{\sinh\alpha l - \sin\alpha l}{2}\right) & \left(\frac{\cosh\alpha l + \cos\alpha l}{2}\right) \end{vmatrix} = 0 \tag{36}$$

Expansion and simplification yield identical expression obtained by Avcar [9]:

$$1 + \cosh\alpha l\cos\alpha l = 0 \tag{37}$$

The frequency equation is a transcendental equation with an infinite number of roots α_i for i = 1, 2, 3, …, n.

The first ten roots of the frequency equation for cantilever Euler-Bernoulli beam are:

$$\begin{aligned} \alpha_1 l &= 1.875104 \\ \alpha_2 l &= 4.694091 \\ \alpha_3 l &= 7.854757 \\ \alpha_4 l &= 10.9955407 \\ \alpha_5 l &= 14.13716831 \\ \alpha_6 l &= 17.2787593 \\ \alpha_7 l &= 20.42035225 \\ \alpha_8 l &= 23.5619449 \\ \alpha_9 l &= 26.703537556 \\ \alpha_{10} l &= 29.84513021 \end{aligned} \tag{38}$$

Then,

$$\omega_n = \alpha_n^2\sqrt{\frac{EI}{m}} \tag{39}$$

$$\omega_1 = \alpha_1^2\sqrt{\frac{EI}{m}} = \left(\frac{1.875104}{l}\right)^2\sqrt{\frac{EI}{m}} = \frac{3.516015}{l^2}\sqrt{\frac{EI}{m}} \tag{40}$$

The natural frequencies of cantilever Euler-Bernoulli beams are presented in Table 3 for the first four vibration modes.

Table-3. Natural frequencies of cantilever Euler-Bernoulli beams for the first four vibration modes.

Vibration mode$n$	Avcar [9]	Natural frequency $\left(\times\sqrt{\frac{EI}{ml^4}}\right)\omega_n$
1	3.5160	3.516015011
2	22.0345	22.03449032
3	61.69721	61.69720753
4	120.9019	120.9019153

D. Euler-Bernoulli beam with clamped-simply supported ends

For an Euler-Bernoulli beam clamped at x = 0 and simply supported at x = l as shown in Figure-4, the boundary conditions are:

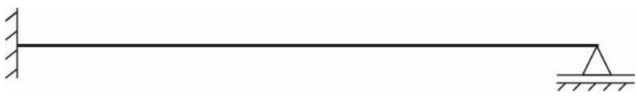

Figure-4. Euler-Bernoulli beam clamped at x = 0 and simply supported at x = l.

$$V(0) = \theta(0) = V'(0) = 0 \tag{41}$$

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$$V(l) = V''(l) = 0 \quad (42)$$

Application of the boundary conditions give the system of homogenous equations:

$$\begin{pmatrix} \frac{\cosh \alpha l - \cos \alpha l}{2\alpha^2} & \frac{\sinh \alpha l - \sin \alpha l}{2\alpha^3} \\ \frac{\cosh \alpha l + \cos \alpha l}{2} & \frac{\sinh \alpha l + \sin \alpha l}{2\alpha} \end{pmatrix} \begin{pmatrix} V''(0) \\ V'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (43)$$

The frequency equation for nontrivial solutions is obtained as:

$$\begin{vmatrix} \frac{\cosh \alpha l - \cos \alpha l}{2\alpha^2} & \frac{\sinh \alpha l - \sin \alpha l}{2\alpha^3} \\ \frac{\cosh \alpha l + \cos \alpha l}{2} & \frac{\sinh \alpha l + \sin \alpha l}{2\alpha} \end{vmatrix} = 0 \quad (44)$$

Expansion and simplification gives the transcendental equation which is identical with results obtained by Avcar [9]:

$$\tan \alpha l = \tanh \alpha l \quad (45)$$

The first four roots of the frequency equation are:

$$\left.\begin{aligned} \alpha_1 l &= \pm 3.92660231204792 \\ \alpha_2 l &= \pm 7.06858274562873 \\ \alpha_3 l &= \pm 10.2101761228130 \\ \alpha_4 l &= \pm 13.3517687777541 \end{aligned}\right\} \quad (46)$$

Then $\omega_n = \alpha_n^2 \sqrt{\frac{EI}{m}}$

For n> 4, $\omega_n = \frac{(n\pi/4)^2}{l^2}\sqrt{\frac{EI}{m}}$ (47)

(n = 5, 9, 13, …)

Identical results are obtained for the Euler-Bernoulli beam clamped at $x = l$ and simply supported at $x = 0$. The natural frequencies of clamped - simply (CS) supported and simply supported-clamped (SC) Euler-Bernoulli beam are presented for the first four vibration modes in Table-4.

Table-4. Natural frequencies of Euler-Bernoulli beams simply supported at x = 0, clamped at x = l or clamped at x = 0, simply supported at x = l for the first four vibration modes.

Vibration mode$n$	Avcar [9]	Natural frequency $\left(\times\sqrt{\frac{EI}{ml^4}}\right)$ ω_n
1	15.4182	15.41820572
2	49.9649	49.96486202
3	104.2477	104.2476964
4	178.2697	178.2697293

The solutions for the eigenvalues or dimensionless natural frequencies of prismatic Euler-Bernoulli beams for various boundary conditions are presented in Tables 5-8.

Table-5. Dimensionless natural frequencies for prismatic Euler-Bernoulli beams, Support condition: clamped-free. Frequency equation: $\cosh \alpha l \cos \alpha l + 1 = 0$.

Vibration mode	Avcar [9]; Mukherjee and Gosai [25] α_i	Present study α_i
1	1.8753	1.875104
2	4.6941	4.694091
3	7.8548	7.854757
4	10.9955	10.9955407
5	14.1372	14.13716831

Table-6. Clamped-clamped Euler-Bernoulli beam. Frequency equation: $\cosh \alpha l \cos \alpha l - 1 = 0$.

Vibration mode	Avcar [9];Mukherjee andGosai [25] α_i	Present study α_i
1	4.7301	4.73014
2	7.8532	7.85321
3	10.9956	10.9956
4	14.1372	14.1372

Table-7. Simply supported-simply supported Euler-Bernoulli beam. Frequency equation: $\cos \alpha l = 0$.

Vibration mode	Mukherjee and Gosai [25] α_i	Avcar [9] α_i	Present study α_i
1	3.1420	3.1415	3.1415926
2	6.2840	6.2832	6.28318531
3	9.4250	9.4248	9.42477796
4	12.5670		12.56637061
5	15.7080		15.70796327

Table-8. Clamped-simply supported Euler-Bernoulli beam.Frequency equation: $\tanh \alpha l = \tan \alpha l$.

Vibration mode	Avcar [9] α_i	Present study α_i
1	3.9266	3.926602
2	7.0686	7.0685827
3	10.2102	10.2101761
4		13.351768778

5. DISCUSSIONS

In this study, the Sumudu transform method (STM) is used to solve the free vibration problem of Euler-Bernoulli beams, represented mathematically by the fourth order partial differential equation (PDE) - Equation (1). Harmonic vibrations are assumed, and the unknown dynamic transverse displacement $v(x, t)$ becomes expressible in coordinate variable – separable form shown in Equation (2). The governing PDE to be solved then simplifies to the fourth order ordinary differential equation (ODE) - Equation (4) or (5).

Application of the Sumudu transform to the ODE results in the integral equation given by Equation (7). Evaluation of the integral equation gave the algebraic problem in Equation (8) which is in terms of $V(u)$ the Sumudu transform of the unknown dynamic modal shape function $V(x)$. Equation (8) is solved for $V(u)$ to obtain $V(u)$ as given by Equation (11). Inverse Sumudu transformation of $V(u)$ is used to obtain the dynamic modal shape function in the physical space variable $V(x)$ as Equation (13). $V(x)$ is expressed in Equation (13) in terms of the initial values $V(0)$, $V'(0)$, $V''(0)$ and $V'''(0)$ which depend upon the end support conditions.

For simply supported ends the Dirichlet boundary conditions given by Equations (14) and (15) are used to obtain the system of homogeneous equations given in matrix form as Equation (16). The characteristic frequency equation for nontrivial solutions obtained from the vanishing of the determinant of the coefficient matrix is found as Equation (17). Expansion of the determinant in Equation (17) gives the characteristic frequency equation as the solvable transcendental equation - Equation (18), which is solved to obtain the natural frequencies as Equation (21) which is presented for the first four vibration modes in Table-1.

Similarly, for Euler-Bernoulli beam with clamped ends, the boundary conditions expressed by Equations (22) and (23) are used to obtain the system of homogeneous equations - Equation (24). For nontrivial solutions, the characteristic frequency equation, obtained from the vanishing of the coefficient matrix is found as Equation (25). Expansion of the determinant gave the characteristic frequency equation as the transcendental equation - Equation (26). Solution of the transcendental equation using the Symbolic Algebra Software, Mathematica Software or other numerical analysis computational tools gave the eigenvalues (roots and zeros) as Equation (27). The natural frequencies are then found for the clamped Euler-Bernoulli beam as Equation (28) and presented for the first four vibration modes in Table-2.

For Euler-Bernoulli beams clamped at $x = 0$ and free at $x = l$, the boundary conditions given by Equations (29) - (34) are used to obtain the system of homogeneous equations - Equation (35). The characteristic frequency equation is obtained as Equation (36) which simplified upon expansion of the determinant to the transcendental equation - Equation (37). The transcendental equation is solved by computational software to obtain the first ten eigenvalues as Equation (38). The natural frequencies of cantilever Euler-Bernoulli beams for the first four vibration modes are presented in Table-3. For Euler-Bernoulli beams clamped at $x = 0$ and simply supported at $x = l$, the boundary conditions given by Equations (41) and (42) are used to obtain the system of homogeneous equations given in matrix form as Equation (43). The resulting characteristic frequency equation is given by Equation (44), which upon expansion and simplification gave the transcendental equation - Equation (45). The first four eigenvalues of the transcendental equation are found as Equation (46) and the natural frequencies are presented in Table-4 for the first four vibration modes. Identical results are obtained for the Euler-Bernoulli beam clamped at $x = l$ and simply supported at $x = 0$.

6. CONCLUSIONS

In conclusion:

a) The Sumudu transform method (STM) has been successfully used to solve the fourth order PDE for the free harmonic vibration of Euler-Bernoulli beams for the following boundary conditions:

a) simply supported ends ($x = 0$, $x = l$)

b) clamped ends ($x = 0$, $x = l$)

c) clamped at $x = 0$, and free at $x = l$

d) clamped at $x = 0$, and simply supported at $x = l$

e) simply supported at $x = 0$, and clamped at $x = l$.

b) The STM converts the boundary value problem (BVP) to an integral equation which is solved to obtain an algebraic problem.

c) The general solution of the problem is obtained in terms of four unknown parameters which are initial values: $V(0)$, $V'(0)$, $V''(0)$ and $V'''(0)$.

d) Enforcement of boundary conditions for the specific cases of end supports considered are used to obtain the corresponding eigenvalue problem represented by a system of homogeneous equations.

e) Conditions for nontrivial solutions which are given by the vanishing of the determinant of the coefficient matrix in the system of homogeneous equations are used to obtain the characteristic frequency equation.

f) The characteristic frequency equations are transcendental equations which are difficult to solve in closed form but can be solved using iteration methods and other approximate numerical methods. The transcendental equations are solved using computational software tools to obtain the eigenvalues (zeros or roots) which are then used to obtain the corresponding natural frequencies.

g) The characteristic frequency equations have an infinite number of roots (eigenvalues) and hence can be used to determine all the natural frequencies of vibration of the Euler-Bernoulli beam. Hence closed form solutions are obtained using the STM adopted in this work.

h) The closed form solutions obtained are identical with previously obtained solutions using the product method and Eigen function expansion methods.

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