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MATHEMATICAL STUDY OF COMPETITIVE PREYS - A PREDATOR WITH HOLLING TYPE-II FUNCTIONAL RESPONSE

Suresh Kumar Y[1] and Seshagiri Rao. N[2]
[1]Department of Mathematics, Koneru Lakshmaiah Education Foundation, Aziznagar, Hyderabad, Telangana State, India
[2]Department of Applied Mathematics, School of Applied Natural Science, Adama Science and Technology University, Adama, Ethiopia
E-Mail: suresh.mscf@gmail.com

ABSTRACT

The present study proposes and explores a food chain model to study about the dynamical behavior of two preys and a predator ecosystem where the preys having competitive interaction among themselves. A Holling type-II functional response has been adopted for first prey and a predator of the proposed model. The boundedness, stability, existence condition of equilibrium of the model is investigated both from analytical and numerical point of view. Hopf bifurcation analysis is also discussed at the positive equilibrium point and a global property of dynamical system is one of the parts of the study. Here, the presented work also utilized to calculate the instability of the population throughout the co-existence's state of steady because of the white noise. At last, the study is hold up by performing the numerical illustrations.

Keywords: holling type-II response function; two preys - predator; local stability; numerical simulations; global stability.

1. INTRODUCTION

In daily routine life, various equations are playing a significant role to calculate the implementation of new methods and ideas. The development of these calculations is really applied to give the direction for various phenomena which can able to demonstrate the various equation that is in the form of equation language. Based on the physical phenomena and the conditions given many of the models are converted as a system of nonlinear differential equations. Later finding the solutions of those models are very difficult by following the analytic techniques so that such complications can be avoidby using either qualitative or numerical techniques which are more effective than analytical techniques. Many biological models have been modeled in terms of nonlinear differential equations. After the pioneer work from the Lotka and Volterra most of the biologists and mathematicians together extended their work under certain conditions around to different types interactions among two species [3, 9, 17, 18, 21] and the multi species [4, 20]. Stability analysis of prey-predator models and their role in environment can observe in [2, 13]. Later many authors have introduced time delay effect to the species. The general discussions on delayed biological systems had in the articles of Cushing [1], May [12], Gopalsamy [5, 6], Martin and Ruan [11], Kunal Chakraborty *et al* [8] and recently Papa Rao [15] with three species system. Hopf bifurcation analysis of three species with or without time delay can be found in [10, 22, 23]. Three species ecological models with different types of functional response among species can see in [7, 14, 16, 19].

The dynamical interaction between three species with Holling type-II response function is included for the first prey and a predator species for the logistical model, whereas the predator species is going to extent is discussed. In this paper we studied the local and global stability of the system at each existing equilibrium points by perturbed technique. Further in the last section we have given the numerical solutions of the system at particular parameter values in the model which interns shows how the system transformed to stability to unstable or vice versa.

2. MATHEMATICAL MODEL

$x(t)$, $y(t)$ are the two prey species populations (in thousands, or millions, or whatever) and $z(t)$ denotes the density of the predators at any instant of time t. Based on the response of Holling type-II functional, the first prey species x is intake by the hunter z and furthermore, both the preys' species (x, y) are competing each other.

2.1 Assumptions

- The parameters r and s are intrinsic growth rates of the prey species x and y respectively
- The prey specie's development is considered as a logistic.
- The parameters k and l are the carrying capacities of two prey species x and y respectively.
- The competing participants for prey species x are defined as the parameter α
- Also, the for-prey species y is defined as the parameter δ.
- Here, the participants are decided based on the competition interaction between both of them.
- The single hunter z 's attack rate which is hunt for the single prey species is considered as the parameter β_1.

- This action is performed whenever the hunter is not currently consuming any prey species.
- The hunter z 's half saturation level is defined as the parameter r_1 which is over the first prey species x.
- The hunter z 's coefficient of natural death rate is denoted as the parameter β_0.
- The first prey x's grazing functional response is represented by the condition of $\frac{\beta_2 xz}{r_1 + x}$ and it is performed by the hunter.
- The Holling type-II functional response is known by this grazing functional response.
- Which is representing the consumption rate of the prey species x by the hunter z.
- Here, Holling type-II's half saturation constant is denoted as r_1.

Here, two preys are aggressively interacted each other where the multi-interaction's model equation is performed between three preys. According to the Holling type-II functional response, the first prey is consumed by the mortal hunter which is derived in the below given formula that is denoting the non-linear decoupled variational equation.

$$\begin{aligned}
\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\alpha xy-\frac{\beta_1 xz}{r_1+x}\\
\frac{dy}{dt} &= sy\left(1-\frac{y}{l}\right)-\delta xy\\
\frac{dz}{dt} &= \frac{\beta_2 xz}{r_1+x}-\beta_0 z
\end{aligned} \tag{2.1}$$

3. BOUNDED AND DISSIPATIVENESS OF THE MODEL

Consider $\Omega_0 = \{(x,y,z)/x,y,z \ge 0\}$. The model equation (2.1) is studied in Ω_0 or in $\overline{\Omega_0}$ for the biological meaning of practical. From the first two equations of the system (2.1), it is easy to derive $\lim sup_{t\to+\infty} x(t) \le k$ and $\lim sup_{t\to+\infty} y(t) \le l$.

Lemma 3.1 The model equation (2.1)'s result $(x(t), y(t), z(t))$ that have the initial values $x(0)\ge 0, y(0)\ge 0, z(0)\ge 0$ which is bounded, dissipative and positive for entire $t \ge 0$ and it is provided as $\beta_2 < \beta_1$.

Proof: Obviously, the solution $(x(t), y(t), z(t))$ of the system (2.1) with initial conditions $x(0)\ge 0, y(0)\ge 0, z(0)\ge 0$ is positive for all $t \ge 0$. Define the function $W(x,y,z) = x+y+z$ then from system (2.1), it follows that

$$\begin{aligned}
\frac{dW}{dt} &= \frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}\\
\frac{dW}{dt} &= \left(rx-\frac{rx^2}{k}-\alpha xy-\frac{\beta_1 xz}{r_1+x}\right)+\left(sy-\frac{sy^2}{l}-\delta xy\right)+\left(\frac{\beta_1 xz}{r_1+x}-\beta_0 z\right)\\
&= rx-\frac{rx^2}{k}-(\alpha+\delta)xy+sy-\frac{sy^2}{l}-\frac{xz}{r_1+x}(\beta_1-\beta_2)-\beta_0 z
\end{aligned}$$

Assume that $\beta_2 < \beta_1$, then the above equation becomes

$$\frac{dW}{dt} \le rx-\frac{rx^2}{k}+sy-\frac{sy^2}{l} \le \frac{r}{k}\left(2kx-x^2\right)+\frac{s}{l}\left(2ly-y^2\right)$$

By introducing positive constant $\xi = \min\{k,l\}$, The aforementioned mathematical formula can derive as given below:

$$\frac{dW}{dt}+\xi W \le \frac{r}{k}\left(k^2-(x-k)^2\right)+\frac{s}{l}\left(l^2-(y-l)^2\right) \le rk+sl = u$$

$$\frac{dW}{dt}+\xi W \le u \Rightarrow W = \frac{u}{\xi}+me^{-\xi t}, \text{ where } m = W(0)-\frac{u}{\xi}$$

$$\Rightarrow W(x(t),y(t),z(t)) \le \frac{u}{\xi}(1-e^{-\xi t})+W(0)e^{-\xi t}$$

Therefore $0 < W(t) \le \frac{u}{\xi}$, for t sufficiently large, provided $\beta_2 < \beta_1$.

Therefore, R_+^3 are confined in the region $\Omega_0 = \left\{(x,y,z)\in R_+^3 : W = x+y+z \le \frac{u}{\xi}\right\}$ which is initiated by the entire results of the model equation (2.1).

VOL. 17, NO. 10, MAY 2022 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

Hence the system is dissipative, provided $\beta_2 < \beta_1$, which completes the proof.

4. EQUILIBRIUM POINTS

The following are the possible equilibrium points which will useful to know the stability of the system (2.1).

- All species extinct state $E_1 : (0,0,0)$
- The equilibrium point $E_2 : (0,l,0)$ on the boundary of second octant.
- The equilibrium point $E_3 : (k,0,0)$ on the boundary of first octant.
- The planner equilibrium $E_4 : (\bar{x}, \bar{y}, 0)$ on the plane $x-y$, where $\bar{x} = \dfrac{ks(r-l\alpha)}{rs-kl\alpha\delta}$ and $\bar{y} = \dfrac{rl(s-k\delta)}{rs-kl\alpha\delta}$.
- The equilibrium points of the boundary $E_5 : (\bar{x}, 0, \bar{z})$ on the axis plane $x-z$, here $\bar{x} = \dfrac{r_1\beta_0}{\beta_2-\beta_0}$ and $\bar{z} = \dfrac{rr_1\beta_2}{\beta_1(\beta_2-\beta_0)}\left(1-\dfrac{kr_1\beta_0}{\beta_2-\beta_0}\right)$.
- The positive equilibrium point $E_6 : (\bar{x}, \bar{y}, \bar{z})$, where $\bar{x} = \dfrac{r_1\beta_0}{\beta_2-\beta_0}$, $\bar{y} = l\left(1-\dfrac{\delta r_1\beta_0}{s(\beta_2-\beta_0)}\right)$ an $\bar{z} = \dfrac{r_1\beta_2}{\beta_1(\beta_2-\beta_0)}\left[r-l\alpha+\dfrac{r_1\beta_0}{\beta_2-\beta_0}\left(\dfrac{l\alpha\delta}{s}-\dfrac{r}{k}\right)\right]$

Now we will see the behavior of the species based on the solutions and nature of the phase plane by computing the eigen values of the 3 X 3 coefficient matrix around each equilibrium of the nonlinear system after reducing into linear system. The three-dimensional linear system has three eigen values at every particular equilibrium point, so by observing the nature of the eigen values we can identify the species behavior near the equilibrium point. i.e. the local stability of the system can have based on the eigen values.

5. EXISTENCE AND STABILITY OF EQUILIBRIUM POINTS

The matrix representation of the linearized system of equations by introducing the small perturbation U such that $X = \bar{X} + U$ over the equilibrium state is as follows:

$$\frac{dU}{dt} = JU$$

where $U = (u_1, u_2, u_3)^T$, $X = (x, y, z)^T$ and

$$J = \begin{bmatrix} r - \dfrac{2r\bar{x}}{k} - \alpha\bar{y} - \beta_1\bar{z}\left(\dfrac{r_1}{(r_1+\bar{x})^2}\right) & -\alpha\bar{x} & -\dfrac{\beta_1\bar{x}}{r_1+\bar{x}} \\ -\delta\bar{y} & s - \dfrac{2s\bar{y}}{l} - \delta\bar{x} & 0 \\ \dfrac{r_1\beta_2\bar{z}}{(r_1+\bar{x})^2} & 0 & \dfrac{\beta_2\bar{x}}{r_1+\bar{x}} - \beta_0 \end{bmatrix}$$

Theorem: 1 At the extent point of the equilibrium, the model equation is always unbalanced.

In this case, the variational matrix for linearized system at $E_1 : (0,0,0)$ is given by

$$J_{E_1} = \begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -\beta_0 \end{bmatrix}$$

The corresponding eigen values are $r, s, -\beta_0$. The extinct equilibrium region becomes saddle point and hence the given system is unstable always.

Theorem: 2 I f $\alpha > \dfrac{r}{l}$, then dynamical system is stable at the equilibrium point E_2 otherwise is unstable.

Proof: The corresponding variational matrix at E_2 is

VOL. 17, NO. 10, MAY 2022 ISSN 1819-6608
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$$J_{E_2} = \begin{bmatrix} r - l\alpha & 0 & 0 \\ -l\delta & -s & 0 \\ 0 & 0 & -\beta_0 \end{bmatrix}$$

The eigen values for this matrix are $l - \alpha r$, $-s$, $-\beta_0$. All three eigen values are negative if $\alpha > \frac{r}{l}$ and hence the system becomes stable state, otherwise the equilibrium region becomes saddle point and so the system is unstable.

Theorem: 3 If $k > \frac{s}{\delta}$ and $k > \frac{\beta_0 r_1}{\beta_2 - \beta_0}$, then the system is stable at E_3 otherwise is unstable.

Proof: The coefficient matrix for linearized system at this point $E_3 : (k,0,0)$ is

$$J_{E_3} = \begin{bmatrix} -r & -\alpha k & -\frac{\beta_1 k}{r_1 + k} \\ 0 & s - \delta k & 0 \\ 0 & 0 & \frac{\beta_2 k}{r_1 + k} - \beta_0 \end{bmatrix}$$

The characteristic equation for this matrix is

$$(\lambda + r)(\lambda - (s - \delta k))\left(\lambda - \left(\frac{\beta_2 k}{r_1 + k} - \beta_0\right)\right) = 0.$$

$-r$, $s - \delta k$ and $\frac{\beta_2 k}{r_1 + k} - \beta_0$ are represents the eigen values of the system. The system is stable if $k > \frac{s}{\delta}$ and $k > \frac{\beta_0 r_1}{\beta_2 - \beta_0}$ in all other cases the system is unstable.

Theorem: 4 If $\frac{\beta_0}{\beta_2} > \frac{ks(r - \alpha l)}{r(rs - kl\alpha\delta) + ks(r - \alpha l)}$, the boundary steady state $E_4(\bar{x}, \bar{y}, 0)$ is stable.

Proof: Equation to calculate the corresponding Jacobin matrix under the equilibrium region $E_4(\bar{x}, \bar{y}, 0)$ is derived below:

$$J_{E_4} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{33} \end{bmatrix}$$

where

$S_{11} = r - \frac{2r\bar{x}}{k} - \alpha\bar{y}$, $S_{12} = -\alpha\bar{x}$, $S_{13} = -\frac{\beta_1\bar{x}}{r_1 + \bar{x}}$, $S_{21} = -\delta\bar{y}$, $S_{22} = s - \frac{2s\bar{y}}{l} - \delta\bar{x}$ and $S_{33} = \frac{\beta_2\bar{x}}{r + \bar{x}} - \beta_0$.

The characteristic expression for the above matrix is represented below:

$$\left(\lambda^2 - (S_{11} + S_{22})\lambda + (S_{11}S_{22} - S_{12}S_{21})\right)(\lambda - S_{33}) = 0$$

.

If $S_{33} < 0$, $S_{11} + S_{22} = -\left(\frac{r\bar{x}}{k} + \frac{s\bar{y}}{l}\right) < 0$ and

$S_{11}S_{22} - S_{12}S_{21} = \bar{x}\,\bar{y}\left(\frac{rs}{lk} - \alpha\delta\right) > 0$ which implies that $\frac{\beta_0}{\beta_2} > \frac{ks(r - \alpha l)}{r(rs - kl\alpha\delta) + ks(r - \alpha l)}$, the equilibrium point $E_4(\bar{x}, \bar{y}, 0)$ is stable otherwise is unstable.

Theorem: 5 The equilibrium region $E_4(\bar{x}, \bar{y}, 0)$ becomes asymptotically stabilized globally at the interior portion R_+^2 of the plane $x - y$ along with the term which is defined in Theorem 4.

Proof: The system becomes reduced to obtain the subsystem in the interior portion of the $x - y$ plane which is represented below:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \alpha xy = f_1(x, y) \text{ and}$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{l}\right) - \delta xy = f_2(x, y)$$

Here, consider $N(x, y)$ is equal to $\frac{1}{xy}$ and it is describes $N(x, y) > 0, \forall (x, y)$ in interior portion of R_+^2.

$$\nabla.[N(x,y)].\begin{bmatrix}\dfrac{dx}{dt}\\ \dfrac{dy}{dt}\end{bmatrix}=\nabla.\left[\frac{1}{xy}\right].\begin{bmatrix}f_1(x,y)\\ f_2(x,y)\end{bmatrix}=\frac{\partial}{\partial x}\left[\frac{r}{y}\left(1-\frac{x}{k}\right)-\alpha\right]+\frac{\partial}{\partial y}\left[\frac{s}{x}\left(1-\frac{y}{l}\right)-\delta\right]$$

$$=-\left(\frac{r}{yk}+\frac{s}{xl}\right)<0,\ \forall\,(x,y)\in R_+^{\,2}$$

Therefore, no time wise result is obtaining in the plane $x-y$'s interior $R_+^{\,2}$ which is based on the criteria of Bendixson-Dulac. Meanwhile, the entire model equation's results are bonded with each other and the equilibrium point of unique point is considered as E_4 which is placed in the plane $x-y$'s interior $R_+^{\,2}$. Therefore, the equilibrium point $E_4(\bar{x},\ \bar{y},\ 0)$ is asymptotically stabilized in worldwide which is in the interior $R_+^{\,2}$ based on Poincare Bendixson-Dulac theorem.

Theorem: 6 The equilibrium region $E_4(\bar{x},\ \bar{y},\ 0)$ becomes asymptotically stabilized in worldwide in the interior $R_+^{\,2}$ with the consideration of term which is derived in Theorem 4.

Proof: The nonlinear system's Lyapunov function is derived as,

$$V_1(x,y,z)=l_1\left\{x-\bar{x}-\bar{x}\log\left(\frac{x}{\bar{x}}\right)\right\}+l_2\left\{y-\bar{y}-\bar{y}\log\left(\frac{y}{\bar{y}}\right)\right\}$$

The following equations are obtained in the basis of variations with respect to t and the replacing results of x,y,z.

$$\frac{dV_1}{dt}=l_1\left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt}+l_2\left(\frac{y-\bar{y}}{y}\right)\frac{dy}{dt}$$
$$=l_1(x-\bar{x})\left[r\left(1-\frac{x}{k}\right)-\alpha y-\frac{\beta_1 z}{r_1+x}\right]+l_2(y-\bar{y})\left[s-\frac{sy}{l}-\delta x\right]$$

By proper selection of $r=\dfrac{r\bar{x}}{k}+\alpha\bar{y}+\dfrac{\beta_1\bar{z}}{r_1+\bar{x}}$, $s=\dfrac{s\bar{y}}{l}+\delta\bar{x}$ the above equation becomes

$$\frac{dV_1}{dt}=-\frac{rl_1}{k}(x-\bar{x})^2-\alpha l_1(x-\bar{x})(y-\bar{y})-\beta_1 l_1(x-\bar{x})\left(\frac{z}{r_1+x}-\frac{\bar{z}}{r_1+\bar{x}}\right)-\frac{sl_2}{l}(y-\bar{y})^2$$
$$-\delta l_2(x-\bar{x})(y-\bar{y})$$
$$=-\frac{rl_1}{k}(x-\bar{x})^2-\frac{sl_2}{l}(y-\bar{y})^2-(\alpha l_1+\delta l_2)(x-\bar{x})(y-\bar{y})$$
$$-l_1\beta_1 r_1\left(\frac{(x-\bar{x})(z-\bar{z})}{(r_1+x)(r_1+\bar{x})}\right)+\left(\frac{l_1\beta_1 z(x-\bar{x})^2}{(r_1+x)(r_1+\bar{x})}\right)-\left(\frac{l_1\beta_1 x(x-\bar{x})(z-\bar{z})}{(r_1+x)(r_1+\bar{x})}\right)$$
$$=-\left(\frac{rl_1}{k}-\frac{l_1\beta_1 z}{(r_1+x)(r_1+\bar{x})}\right)(x-\bar{x})^2-\frac{sl_2}{l}(y-\bar{y})^2-(\alpha l_1+\delta l_2)(x-\bar{x})(y-\bar{y})$$
$$-l_1\beta_1\left(\frac{(x-\bar{x})(z-\bar{z})}{(r_1+\bar{x})}\right)$$

Choose $l_1=\dfrac{r_1+\bar{x}}{\beta_1}, l_2=1$, the above equation becomes

$$\frac{dV_1}{dt}=-\left(\frac{r(r_1+\bar{x})}{k\beta_1}-\frac{z}{(r_1+x)}\right)(x-\bar{x})^2-\frac{s}{l}(y-\bar{y})^2-\left(\frac{\alpha(r_1+\bar{x})}{\beta_1}+\delta\right)(x-\bar{x})(y-\bar{y})$$
$$-(x-\bar{x})(z-\bar{z})$$
$$=-\left(\frac{r(r_1+\bar{x})}{k\beta_1}-\frac{z}{(r_1+x)}+\frac{1}{2}+\frac{1}{2}\left(\frac{\alpha(r_1+\bar{x})}{\beta_1}+\delta\right)\right)(x-\bar{x})^2$$
$$-\left[\frac{s}{l}+\frac{1}{2}\left(\frac{\alpha(r_1+\bar{x})}{\beta_1}+\delta\right)\right](y-\bar{y})^2-\frac{1}{2}(z-\bar{z})^2$$

$$\frac{dV_1}{dt}=-\left(\frac{r(r_1+\bar{x})}{k\beta_1}-\frac{z}{(r_1+x)}+\frac{1}{2}+\frac{1}{2}\left(\frac{\alpha(r_1+\bar{x})}{\beta_1}+\delta\right)\right)(x-\bar{x})^2$$
$$-\left[\frac{s}{l}+\frac{1}{2}\left(\frac{\alpha(r_1+\bar{x})}{\beta_1}+\delta\right)\right](y-\bar{y})^2-\frac{1}{2}(z-\bar{z})^2$$

VOL. 17, NO. 10, MAY 2022 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

If $\frac{r(r_1+\bar{x})}{k\beta_1}+\frac{1}{2}+\frac{1}{2}\left(\frac{\alpha(r_1+\bar{x})}{\beta_1}+\delta\right)>\frac{z}{(r_1+x)}$ then $\frac{dV_1}{dt}<0$ and hence by known theorem, At the equilibrium region the system becomes globally stable $E_4(\bar{x},\ \bar{y},\ 0)$.

Theorem: 7 If $\bar{x}>\frac{s}{\delta}$ and $\bar{z}<\frac{r}{k\beta_1}(r_1+\bar{x})^2$ then the boundary steady state $E_5\left(\bar{x},\ 0,\bar{z}\right)$ becomes stable in $x-z$ plane.

Proof: For this Theorem, the equivalent Jacobin matrix expression is given below:

$$J_{E_5}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

where

$a_{11}=r-\frac{2r\bar{x}}{k}-\frac{r_1\beta_1\bar{z}}{(r_1+\bar{x})^2}$, $a_{12}=-\alpha\bar{x}$, $a_{13}=-\frac{\beta_1\bar{x}}{r_1+\bar{x}}$, $a_{22}=s-\delta\bar{x}$

and $a_{31}=\frac{r\beta_2\bar{z}}{(r_1+\bar{x})^2}$.

The characteristic equation of J_{E_5} is $\left(\lambda^2-a_{11}\lambda-a_{13}a_{31})\right)\left(a_{22}-\lambda\right)=0$ .One can observe that $a_{22}<0$ and $a_{11}<0$, $a_{13}a_{31}>0$ which implies that $\bar{x}>\frac{s}{\delta}$ and $\bar{z}<\frac{r}{k\beta_1}(r_1+\bar{x})^2$, the equilibrium region $E_5\left(\bar{x},\ 0,\bar{z}\right)$ becomes unstable when it is not lying on the plane $x-z$.

Theorem: 8 The equilibrium region $E_5(\bar{x},\ 0,\bar{z})$ is asymptotically stabilized in worldwide according to theTheorem-7 and $x>(r_1+k)$ which is lying on the plane $x-z$'s interior $R_+{}^2$.

Proof: The model equation (2.1) is decreases to following sub equation for any value in the $x-z$ plane's interior $R_+{}^2$ which is placed in the same plane's interior $R_+{}^2$.

$\frac{dx}{dt}=rx\left(1-\frac{x}{k}\right)-\frac{\beta_1 xz}{r_1+x}=f_3(x,z)and\frac{dz}{dt}=\frac{\beta_2 xz}{r_1+x}-\beta_0 z=f_4(x,z)$

Now, let$M(x,z)=\frac{r_1+x}{xz}$, then$M(x,z)>0,\forall(x,z)$ in the interior of $R_+{}^2$.

$$\nabla.[\mathrm{M(x,z)}].\begin{bmatrix}\frac{dx}{dt}\\ \frac{dz}{dt}\end{bmatrix}=\nabla.\left[\left(\frac{r_1+x}{xz}\right)\begin{pmatrix}f_3(x,z)\\ f_4(x,z)\end{pmatrix}\right]$$
$$=\frac{\partial}{\partial x}\left[r\left(\frac{r_1+x}{z}\right)\left(1-\frac{x}{k}\right)-\beta_1\right]+\frac{\partial}{\partial z}\left[\beta_2-\beta_0\left(\frac{r_1+x}{x}\right)\right]$$
$$=-\frac{r}{z}\left(\frac{x}{k}-1\right)-\frac{r}{k}\left(\frac{r_1+x}{z}\right)$$
$$=-\frac{r}{k}\left(\frac{x-(r_1+k)}{z}\right)<0,\forall(x,z)\in {R_+}^2$$

According to the Bendixson-Dulac conditions, In the interior portion of $R_+{}^2$ of plane $x-z$ does not consists of periodic results. Meanwhile, the entire model equation's results are bonded with each other and the equilibrium point of unique point is considered as E_5 which is placed in the plane $x-z$'s interior $R_+{}^2$. Therefore, the equilibrium region $E_5(\bar{x},\ 0,\bar{z})$ becomes asymptotically stabilized in worldwide which is in the interior $R_+{}^2$ based on Poincare Bendixson-Dulac theorem.

Theorem: 9 When $\frac{\alpha+2r}{2k}>\frac{\beta_1\bar{z}}{(r_1+x)(r_1+\bar{x})}$ the equilibrium point $E_5(\bar{x},\ 0,\bar{z})$ is asymptotically stabilized in worldwide in the interior $R_+{}^3$ with the consideration of term which is derived in Theorem 7.

Proof: The positive definite function is considered as

$$V_2(x,z)=m_1\left[x-\bar{x}-\bar{x}\log\left(\frac{x}{\bar{x}}\right)\right]+m_2\left[z-\bar{z}-\bar{z}\log\left(\frac{z}{\bar{z}}\right)\right]$$

Here, the positive constants are described as m_1,m_2.

The following equations are obtained in the basis of variations with respect to t and the replacing results of x,z.

$$\frac{dV_2}{dt}=m_1\left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt}+m_2\left(\frac{z-\bar{z}}{z}\right)\frac{dz}{dt}$$
$$=m_1(x-\bar{x})\left[r\left(1-\frac{x}{k}\right)-\alpha y-\frac{\beta_1 z}{r_1+x}\right]+m_2(z-\bar{z})\left[\frac{\beta_2 x}{r_1+x}-\beta_0\right]$$

By proper selection of $r=\frac{r\bar{x}}{k}+\alpha\bar{y}+\frac{\beta_1\bar{z}}{r_1+\bar{x}}$, $\beta_0=\frac{\beta_2\bar{x}}{r_1+\bar{x}}$, the above equation becomes

ARPN Journal of Engineering and Applied Sciences

$$\frac{dV_2}{dt}=-\frac{rm_1}{k}(x-\bar{x})^2-\alpha m_1(x-\bar{x})(y-\bar{y})-\beta_1 m_1(x-\bar{x})\left(\frac{z}{r_1+x}-\frac{\bar{z}}{r_1+\bar{x}}\right)$$
$$+m_2\beta_2(z-\bar{z})\left[\frac{x}{r_1+x}-\frac{\bar{x}}{r_1+\bar{x}}\right]$$
$$=-\left[\frac{rm_1}{k}-\frac{\beta_1 m_1\bar{z}}{(r_1+x)(r_1+\bar{x})}\right](x-\bar{x})^2-\alpha m_1(x-\bar{x})(y-\bar{y})-\beta_1 m_1\left(\frac{(x-\bar{x})(z-\bar{z})}{(r_1+x)}\right)$$
$$+\beta_2 m_2 r_1\left[\frac{(x-\bar{x})(z-\bar{z})}{(r_1+x)(r_1+\bar{x})}\right]$$

The aforementioned equation is modified as given below equation by selecting the non-negative constants $m_1=1$ and $m_2=\dfrac{\beta_1(r_1+\bar{x})}{\beta_2 r_1}$.

$$\frac{dV_2}{dt}=-\left[\frac{r}{k}-\frac{\beta_1\bar{z}}{(r_1+x)(r_1+\bar{x})}\right](x-\bar{x})^2-\alpha(x-\bar{x})(y-\bar{y})$$
$$=-\left[\frac{r}{k}-\frac{\beta_1\bar{z}}{(r_1+x)(r_1+\bar{x})}+\frac{\alpha}{2}\right](x-\bar{x})^2-\frac{\alpha}{2}(y-\bar{y})^2$$

$\Rightarrow \dfrac{dV_1}{dt}<0$, if $\dfrac{\alpha+2r}{2k}>\dfrac{\beta_1\bar{z}}{(r_1+x)(r_1+\bar{x})}$

Therefore, if $\dfrac{\alpha+2r}{2k}>\dfrac{\beta_1\bar{z}}{(r_1+x)(r_1+\bar{x})}$, the equilibrium point $E_5(\bar{x},0,\bar{z})$ is asymptotically stabilized in worldwide with the help of the theorem of Lyapunov.

Theorem: 10 The equilibrium region in interior portion $E_6:(\bar{x},\bar{y},\bar{z})$ exists if $\beta_2>\beta_0, r>l\alpha$ and $kl\alpha\delta>rs$.

Proof: assume that $\bar{x}$, $\bar{y}$, $\bar{z}$ are the positive results for the following expression

$$r\left(1-\frac{x}{k}\right)-\alpha y-\frac{\beta_1 z}{r_1+x}=0,\ s\left(1-\frac{y}{l}\right)-\delta x=0$$

and $\dfrac{\beta_2 x}{r_1+x}-\beta_0=0$. By solving above expression we obtain $\bar{x}=\dfrac{r_1\beta_0}{\beta_2-\beta_0}$, $\bar{y}=l\left(1-\dfrac{\delta r_1\beta_0}{s(\beta_2-\beta_0)}\right)$ and

$$\bar{z}=\frac{r_1\beta_2}{\beta_1(\beta_2-\beta_0)}\left[r-l\alpha+\frac{r_1\beta_0}{\beta_2-\beta_0}\left(\frac{l\alpha\delta}{s}-\frac{r}{k}\right)\right].$$

Hence the equilibrium region at interior portion $E_6(\bar{x},\bar{y},\bar{z})$ exists if $\beta_2>\beta_0$, $r>l\alpha$ and $kl\alpha\delta>rs$.

Theorem: 11 If $B_1>0, B_3>0$ and $B_1B_2>B_3$, the interior equilibrium region $E_6(\bar{x},\bar{y},\bar{z})$ is asymptotically stabilized in local wise.

Proof: Jacobian matrix is given below for the equilibrium region condition.

$$J_{E_6}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & 0\\ a_{31} & 0 & 0\end{bmatrix}$$

here

$a_{11}=\dfrac{\beta_1\bar{x}\bar{z}}{(r_1+\bar{x})^2}-\dfrac{r\bar{x}}{k}$, $a_{12}=-\alpha\bar{x}$, $a_{13}=-\dfrac{\beta_1\bar{x}}{r_1+\bar{x}}$, $a_{21}=-\delta\bar{y}$, $a_{22}=-\dfrac{s\bar{y}}{l}$

and $a_{31}=\dfrac{r_1\beta_2\bar{z}}{(r_1+\bar{x})^2}$.

The characteristic equation of J_{E_6} is $\lambda^3+B_1\lambda^2+B_2\lambda+B_3=0$.

Here $B_1=-(a_{11}+a_{12})=\dfrac{N_2}{klN_1^2}$

$B_2=a_{11}a_{22}-a_{12}a_{21}-a_{13}a_{31}$

$B_3=a_{13}a_{31}a_{22}=\dfrac{r_1 s\beta_1\beta_2\bar{x}\,\bar{y}\,\bar{z}}{lN_1^3}$ and

$N_1=(r_1+\bar{x})>0, N_2=r\bar{x}lN_1^2+s\bar{y}kN_1^2-kl\beta_1\overline{xz}>0$,

when $\beta_1<\dfrac{(rl\bar{x}+ks\bar{y})N_1^2}{kl\overline{xz}}$.

If $\beta_1<\dfrac{(rl\bar{x}+ks\bar{y})N_1^2}{kl\overline{xz}}$ the coefficients of characteristic equation $B_1>0$ and $B_3>0$.

Now

$$\Delta=B_1B_2-B_3=-(a_{11}+a_{12})(a_{11}a_{22}-a_{12}a_{21})+a_{11}a_{13}a_{31}$$
$$\Delta=-\frac{\beta_1 s\bar{x}\,\bar{y}\,\bar{z}\,N_2}{kl^2N_1^4}+\frac{\bar{x}\,\bar{y}\,N_2}{klN_1^2}\left(\frac{rs}{kl}-\alpha\delta\right)+\frac{r_1\beta_1\beta_2\bar{x}\,\bar{z}}{N_1^3}\left(\frac{r\bar{x}}{k}-\frac{\beta_1\bar{x}\,\bar{z}}{N_1^2}\right)$$

If

$$\beta_2>\frac{N_1^3N_2\bar{y}}{l^2r_1\beta_1\overline{xz}(rN_1^2-k\beta_1\bar{z})}\left[\frac{\beta_1 s\bar{z}}{N_1^2}+\frac{N_2}{k}(kl\alpha\delta-rs)\right]$$

then $\Delta>0$.

The necessary and sufficient conditions for the stability of the system by Routh-Hurwitz criterion is $B_1>0, B_3>0$ and $B_1B_2>B_3$. It is clear from above that it holds the conditions of Routh-Hurwitz criterion at the satisfied conditions and hence the system is locally stable at this equilibrium point otherwise is unstable in all other cases.

Theorem: 12 Along with the conditions stated in the Theorem-11 and if $\dfrac{2r+k(\alpha+\delta\beta_1)}{2k\beta_1}>\dfrac{\bar{z}}{(r_1+x)(r_1+\bar{x})}$, the interior equilibrium point $E_6(\bar{x},\bar{y},\bar{z})$ is globally asymptotically stable.

Proof: Let the Lyapunov function for the nonlinear system be

$$V_3(x,y,z)=n_1\left[x-\bar{x}-\bar{x}\log\left(\frac{x}{\bar{x}}\right)\right]+n_1\left[y-\bar{y}-\bar{y}\log\left(\frac{y}{\bar{y}}\right)\right]+n_3\left[z-\bar{z}-\bar{z}\log\left(\frac{z}{\bar{z}}\right)\right]$$

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ARPN Journal of Engineering and Applied Sciences

The following equations are obtained in the basis of variations with respect to t and the replacing results of x, y, z.

$$\frac{dV_3}{dt} = n_1\left(\frac{x-\bar{x}}{x}\right)\frac{dx}{dt} + n_2\left(\frac{y-\bar{y}}{y}\right)\frac{dy}{dt} + n_3\left(\frac{z-\bar{z}}{z}\right)\frac{dz}{dt}$$
$$= n_1(x-\bar{x})\left[r\left(1-\frac{x}{k}\right) - \alpha y - \frac{\beta_1 z}{r_1+x}\right] + n_2(y-\bar{y})\left[s - \frac{sy}{l} - \delta x\right]$$
$$+ n_3(z-\bar{z})\left[\frac{\beta_2 x}{r_1+x} - \beta_0\right]$$

By proper selection of $r = \frac{r\bar{x}}{k} + \alpha\bar{y} + \frac{\beta_1\bar{z}}{r_1+\bar{x}}$, $s = \frac{s\bar{y}}{l} + \delta\bar{x}$, $\beta_0 = \frac{\beta_2\bar{x}}{r_1+\bar{x}}$, the above equation becomes

$$\frac{dV_3}{dt} = n_1(x-\bar{x})\left[-\frac{r}{k}(x-\bar{x}) - \alpha(y-\bar{y}) - \beta_1\left(\frac{z}{r_1+x} - \frac{\bar{z}}{r_1+\bar{x}}\right)\right]$$
$$+ n_2(y-\bar{y})\left[-\frac{s}{l}(y-\bar{y}) - \delta(x-\bar{x})\right] + n_3(z-\bar{z})\beta_2\left[\frac{x}{r_1+x} - \frac{\bar{x}}{r_1+\bar{x}}\right]$$
$$= -\frac{rn_1}{k}(x-\bar{x})^2 - \frac{sn_2}{l}(y-\bar{y})^2 - (\alpha n_1 + \delta n_2)(x-\bar{x})(y-\bar{y})$$
$$-\beta_1 n_1(x-\bar{x})\left(\frac{r_1(z-\bar{z}) + \bar{x}(z-\bar{z}) - \bar{z}(x-\bar{x})}{(r_1+x)(r_1+\bar{x})}\right) + r_1\beta_2 n_3$$
$$\frac{dV_3}{dt} = -\frac{rn_1}{k}(x-\bar{x})^2 - \frac{sn_2}{l}(y-\bar{y})^2 - (\alpha n_1 + \delta n_2)(x-\bar{x})(y-\bar{y})$$
$$-\beta_1 n_1\left(\frac{(x-\bar{x})(z-\bar{z})}{(r_1+x)}\right) + \beta_1 n_1\bar{z}\left(\frac{(x-\bar{x})^2}{(r_1+x)(r_1+\bar{x})}\right) + r_1\beta_2 n_3\left[\frac{(x-\bar{x})(z-\bar{z})}{(r_1+x)(r_1+\bar{x})}\right]$$
$$= -\left[\frac{rn_1}{k} - \frac{\beta_1 n_1\bar{z}}{(r_1+x)(r_1+\bar{x})}\right](x-\bar{x})^2 - \frac{sn_2}{l}(y-\bar{y})^2 - (\alpha n_1 + \delta n_2)(x-\bar{x})(y-\bar{y})$$
$$-\beta_1 n_1\left(\frac{(x-\bar{x})(z-\bar{z})}{(r_1+x)}\right) + r_1\beta_2 n_3\left[\frac{(x-\bar{x})(z-\bar{z})}{(r_1+x)(r_1+\bar{x})}\right]$$

By choosing $n_1 = \frac{1}{\beta_1}, n_2 = 1$ and $n_3 = \frac{r_1+\bar{x}}{\beta_2 r_1}$, the above equation becomes,

$$\frac{dV_3}{dt} = -\left[\frac{r}{k\beta_1} - \frac{\bar{z}}{(r_1+x)(r_1+\bar{x})}\right](x-\bar{x})^2 - \frac{s}{l}(y-\bar{y})^2 - \left(\frac{\alpha}{\beta_1} + \delta\right)(x-\bar{x})(y-\bar{y})$$
$$= -\left[\frac{r}{k\beta_1} - \frac{\bar{z}}{(r_1+x)(r_1+\bar{x})} + \frac{1}{2}\left(\frac{\alpha}{\beta_1} + \delta\right)\right](x-\bar{x})^2 - \left[\frac{s}{l} + \frac{1}{2}\left(\frac{\alpha}{\beta_1} + \delta\right)\right](y-\bar{y})^2$$

If $\frac{2r + k(\alpha + \delta\beta_1)}{2k\beta_1} > \frac{\bar{z}}{(r_1+x)(r_1+\bar{x})}$

then $\frac{dV_3}{dt} < 0$ and hence from known according to the theorem, equilibrium region in interior portion $E_6(\bar{x}, \bar{y}, \bar{z})$ is on globally asymptotically stable state.

6. HOPF BIFURCATION

In this study, the performance of the dynamic system is showcased by utilizing many parameters. Whenever the species are co-existing in equilibrium state, the constant prey-predator parameters are often established on steady state. Even though, if any changes are takes place in the parameter of the model the entire performance of the system can be changed. Bifurcation points is a transition which is occurs due to the condemnatory parameter values. Whenever, a system has significant periodic results at that time Hopf bifurcation occurs. In this portion discussed about Hopf bifurcation occurs at the point of condemnatory value $\beta_2 = \beta_2^*$

Theorem 6.1 consider that $\beta_1 < \frac{(rl\bar{x} + ks\bar{y})N_1^2}{kl\bar{x}\bar{z}}$, at the point of condemnatory value a uncomplicated Hopf bifurcation occurs for the system. the condemnatory value is $\beta_2 = \beta_2^*$.

Proof: Liu technique is utilized for examining the model of the Hopf bifurcation.

Consider that local stability constraints are taken for the Hopf bifurcation model

$$\beta_2^* = \frac{N_1^3 N_2\bar{y}}{l^2 r_1\beta_1\bar{x}\bar{z}(rN_1^2 - k\beta_1\bar{z})}\left[\frac{\beta_1 s\bar{z}}{N_1^2} + \frac{N_2}{k}(kl\alpha\delta - rs)\right],$$

then

$$B_1\big|_{\beta_2=\beta_2^*} = \frac{N_2}{klN_1^2} > 0$$

$$B_3\big|_{\beta_2=\beta_2^*} = \frac{N_2 s\bar{y}^2}{l^3(rN_1^2 - k\beta_1\bar{z})}\left[\frac{\beta_1 s\bar{z}}{N_1^2} + \frac{N_2}{k}(kl\alpha\delta - rs)\right] > 0$$

$$\left.\frac{d\Delta}{d\beta_2}\right|_{\beta_2=\beta_2^*} = \frac{r\beta_1\bar{x}^2\bar{z}}{kN_1^5}(rN_1^2 - k\beta_1\bar{z}) \neq 0$$

Therefore $\left.\frac{d\Delta}{d\beta_2}\right|_{\beta_2=\beta_2^*} \neq 0$. Thus, uncomplicated Hopf Bifurcation occurs at the point of the condemnatory value of $\beta_2 = \beta_2^*$ is proved

7. STOCHASTIC ANALYSIS

In this portion, developed the stochastic type of framework (2.1) which is utilized for identifying the impact of the noise. By the stochastic type of framework, enumerated the variances occupied on the positive equilibrium state owing to the noise. The additive gaussian white noise occurs at the model due to the irregular noise of the model and their disturbances described below:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right) - \alpha xy - \frac{\beta_1 xz}{r_1+x} + k_1\xi_1(t)$$
$$\frac{dy}{dt} = sy\left(1-\frac{y}{l}\right) - \delta xy + k_2\xi_2(t) \qquad (7.1)$$
$$\frac{dz}{dt} = \frac{\beta_2 xz}{r_1+x} - \beta_0 z + k_3\xi_3(t)$$

ARPN Journal of Engineering and Applied Sciences

Here two preys terms are denoted as x(t), y(t) and hunter terms are z(t).

k_1, k_2, k_3 are assumed as a real constant. $\xi(t) = \left(\xi_1(t), \xi_2(t), \xi_3(t)\right)$ represents the three-dimension procedure of the Gaussian white noise which is utilized for satisfying $E\left[\xi_i(t)\right] = 0,\ i = 1,2,3$. $E\left[\xi_i(t)\xi_j(t')\right] = \delta_{ij}\,\delta(t-t')$, $i = j = 1,2,3$.

Kronecker symbol is denoted as δ_{ij}, Dirac operation is denoted as δ.

Let $x(t) = u_1 + T^*, y(t) = u_2 + S^*, z(t) = u_3 + R^*$

The species processing time derivatives are given below:

$\frac{dx}{dt} = \frac{du_1}{dt}, \frac{dy}{dt} = \frac{du_2}{dt}, \frac{dz}{dt} = \frac{du_3}{dt}$, replacing the above derivatives in expression (7.1)

$$\frac{du_1}{dt} = r(u_1+T^*)\left(1-\frac{u_1+T^*}{k}\right) - \alpha(u_1+T^*)(u_2+S^*) - \frac{\beta_1(u_1+T^*)(u_3+R^*)}{r_1+(u_1+T^*)} + k_1\xi_1(t)$$
$$\frac{du_2}{dt} = s(u_2+S^*)\left(1-\frac{u_2+S^*}{l}\right) - \delta(u_1+T^*)(u_2+S^*) + k_2\xi_2(t)$$
$$\frac{du_3}{dt} = \frac{\beta_2(u_1+T^*)(u_3+R^*)}{r_1+(u_1+T^*)} - \beta_0(u_3+R^*) + k_3\xi_3(t) \tag{7.2}$$

Linear part of the above equation is

$$\frac{du_1}{dt} = -\frac{r}{k}u_1T^* - \alpha u_2T^* - \frac{\beta_1}{r_1}u_3T^* + k_1\xi_1(t)$$
$$\frac{du_2}{dt} = -\delta u_1S^* - \frac{s}{l}u_2S^* + k_2\xi_2(t)$$
$$\frac{du_3}{dt} = \frac{\beta_2}{r_1}u_1R^* + k_3\xi_3(t) \tag{7.3}$$

By taking fourier transform to the expression of (7.3) on the both sides, we obtain

$$k_1\tilde{\xi}_1(\omega) = \left(i\omega + \frac{r}{k}T^*\right)\tilde{u}_1(\omega) + \alpha T^*\tilde{u}_2(\omega) + \frac{\beta_1}{r_1}T^*\tilde{u}_3(\omega)$$
$$k_2\tilde{\xi}_2(\omega) = \delta S^*\tilde{u}_1(\omega) + \left(i\omega + \frac{s}{l}S^*\right)\tilde{u}_2(\omega)$$
$$k_3\tilde{\xi}_3(\omega) = -\frac{\beta_2}{r_1}R^*\tilde{u}_1(\omega) + i\omega\tilde{u}_3(\omega) \tag{7.4}$$

The following expression (7.5) represents the matrix form of the above expression (7.4)

$$M(\omega)\tilde{u}(\omega) = \tilde{\xi}(\omega) \tag{7.5}$$

where $M(\omega) = \begin{bmatrix} A(\omega) & B(\omega) & C(\omega) \\ D(\omega) & E(\omega) & F(\omega) \\ G(\omega) & H(\omega) & I(\omega) \end{bmatrix}; \tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \\ \tilde{u}_3(\omega) \end{bmatrix}; \tilde{\xi}(\omega) = \begin{bmatrix} \tilde{\xi}_1(\omega) \\ \tilde{\xi}_2(\omega) \\ \tilde{\xi}_3(\omega) \end{bmatrix}$

and $A(\omega) = i\omega + \frac{r}{k}T^*; B(\omega) = \alpha T^*; C(\omega) = \frac{\beta_1}{r_1}T^*$

$D(\omega) = \delta S^*; E(\omega) = i\omega + \frac{s}{l}S^*; F(\omega) = 0$

$G(\omega) = -\frac{\beta_2}{r_1}R^*; H(\omega) = 0; I(\omega) = i\omega$

From the expression (7.5), we obtain

$$\tilde{u}(\omega) = [M(\omega)]^{-1}\tilde{\xi}(\omega) \tag{7.6}$$

Let $\left[M(\omega)\right]^{-1} = L(\omega)$, where

$$L(\omega) = \frac{Adj M(\omega)}{|M(\omega)|} \tag{7.7}$$

$$\text{Therefore}\ \tilde{u}(\omega) = L(\omega)\tilde{\xi}(\omega) \tag{7.8}$$

Whether Y(t) function contains a mean value of zero then the variances that is fluctuation intensity of the function components with the ranges of frequency $(\omega, \omega + d\omega)$ is considered as $S_Y(\omega)d\omega$. Where $S_Y(\omega$ denotes the function Y spectral density and which is expressed mathematical by the following expression

$$S_Y(\omega) = \underset{\tilde{T}\to\infty}{Lt}\frac{|\tilde{Y}(\omega)|^2}{\tilde{T}} \tag{7.9}$$

Whether the function Y contains a mean value of zero then the invertible transform of $S_Y(\omega)$ becomes auto covariance operation

$$C_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)e^{i\omega\tau}d\omega \tag{7.10}$$

The fluctuation intensity of the function Y(t) is described in mathematical expression which is given below

$$\sigma_Y^2 = C_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega \tag{7.11}$$

Next, auto correlation operation is transformed into function named auto variance

$$P_Y(\tau) = \frac{C_Y(\tau)}{C_Y(0)} \tag{7.12}$$

The mathematical expression for the Gaussian White noise procedure is expressed below:

$= \underset{\hat{T}\to+\infty}{Lt} \frac{E[\xi_i(\omega)\xi_j(\omega)]}{\hat{T}} =$

$\underset{\hat{T}\to+\infty}{Lt} \frac{1}{\hat{T}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} E[\xi_i(\omega)\xi_j(\omega)] S_{\xi_i\xi_j} e^{-i\omega(t-t')} dt dt' = \delta_{ij}$ (7.13)

By the expression (7.6), we obtain

$\tilde{u}(\omega) = \sum_{j=1}^{3} L_{ij}(\omega)\xi_j(\omega); i = 1,2,3$ (7.14)

By (7.9), we have

$$S_{u_i}(\omega) = \sum_{j=1}^{3} \alpha_j(\omega)\left|L_{ij}(\omega)\right|^2 ; i = 1,2,3$$

By (7.7), we obtain

$$\sigma_{u_1}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} k_1 \left|\frac{A_1}{|M(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_2 \left|\frac{B_1}{|M(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_3 \left|\frac{C_1}{|M(\omega)|}\right|^2 d\omega\right\}$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} k_1 \left|\frac{A_2}{|M(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_2 \left|\frac{B_2}{|M(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_3 \left|\frac{C_2}{|M(\omega)|}\right|^2 d\omega\right\}$$

$$\sigma_{u_3}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} k_1 \left|\frac{A_3}{|M(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_2 \left|\frac{B_3}{|M(\omega)|}\right|^2 d\omega + \int_{-\infty}^{\infty} k_3 \left|\frac{C_3}{|M(\omega)|}\right|^2 d\omega\right\} \quad (7.15)$$

where

$\left|M(\omega)\right| = \left|R(\omega)\right| + i\left|I(\omega)\right|$, the real and imaginary parts of $\left|M(\omega)\right|$ are

$$R^2(\omega) = -\frac{\omega^2 sS^*}{l} - \frac{\omega^2 rT^*}{k} + \frac{\beta_1\beta_2 sT^*R^*S^*}{lr_1^2}$$

$$I^2(\omega) = -\omega^3 + \frac{\omega rsT^*S^*}{kl} - \omega\alpha\delta T^*S^* + \frac{\omega\beta_1\beta_2 T^*R^*}{r_1^2}$$

And

$|A_1|^2 = X_1^2 + Y_1^2; |B_1|^2 = X_2^2 + Y_2^2; |C_1|^2 = X_3^2 + Y_3^2$

$|A_2|^2 = X_4^2 + Y_4^2; |B_2|^2 = X_5^2 + Y_5^2; |C_2|^2 = X_6^2 + Y_6^2$

$|A_3|^2 = X_7^2 + Y_7^2; |B_3|^2 = X_8^2 + Y_8^2; |C_3|^2 = X_9^2 + Y_9^2$

where

$$X_1 = -\omega^2; Y_1 = \frac{\omega sS^*}{l}; X_2 = 0; Y_2 = \omega\alpha T^*; X_3 = -\frac{\beta_1 sT^*S^*}{r_1 l}; Y_3 = -\frac{\omega\beta_1 T^*}{r_1}$$

$$X_4 = 0; Y_4 = \omega\delta S^*; X_5 = \omega^2 - \frac{\beta_1\beta_2 R^*T^*}{r_1^2}; Y_5 = \frac{\omega rT^*}{k}; X_6 = -\frac{\beta_1\delta S^*T^*}{r_1}; Y_6 = 0$$

$$X_7 = \frac{\beta_2 sR^*S^*}{r_1 l}; Y_7 = \frac{\omega\beta_2 R^*}{r_1}; X_8 = \frac{\beta_2\alpha R^*T^*}{r_1}; Y_8 = 0; X_9 = -\omega^2 + \frac{rsS^*T^*}{kl} - \alpha\delta S^*T^*;$$

$$Y_9 = \frac{\omega sS^*}{l} + \frac{\omega rT^*}{k}$$

The equation (7.15) becomes

$$\sigma_{u_1}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} [k_1(X_1^2 + Y_1^2) + k_2(X_2^2 + Y_2^2) + k3(X_3^2 + Y_3^2)]\, d\omega\right\}$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} [k_1(X_4^2 + Y_4^2) + k_2(X_5^2 + Y_5^2) + k3(X_6^2 + Y_6^2)]\, d\omega\right\}$$

$$\sigma_{u_3}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} [k_1(X_7^2 + Y_7^2) + k_2(X_8^2 + Y_8^2) + k3(X_9^2 + Y_9^2)] d\omega\right\}$$

If replacing the values of k_1, k_2, k_3 as zero to obtain the dynamic performance of the system for the expression (7.1), then the population of fluctuation intensity becomes

Case: 1 Here the value of k_1 and k_2 is taken as zero, then

$$\sigma_{u_1}^2 = \frac{k_3(X_3^2 + Y_3^2)}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega\right\}$$

$$\sigma_{u_2}^2 = \frac{k_3(X_6^2 + Y_6^2)}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega\right\}$$

$$\sigma_{u_3}^2 = \frac{k_3(X_9^2 + Y_9^2)}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega\right\}$$

Case: 2 Here the value of k_2 and k_3 is taken as zero, then

$$\sigma_{u_1}^2 = \frac{k_1(X_1^2 + Y_1^2)}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega\right\}$$

$$\sigma_{u_2}^2 = \frac{k_1(X_4^2 + Y_4^2)}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega\right\}$$

$$\sigma_{u_3}^2 = \frac{k_1(X_7^2 + Y_7^2)}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega\right\}$$

Case: 3 Here the value of k_1, and k_3 is taken as zero, then

$$\sigma_{u_1}^2 = \frac{k_2(X_2^2 + Y_2^2)}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega\right\}$$

$$\sigma_{u_2}^2 = \frac{k_2(X_5^2 + Y_5^2)}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega\right\}$$

$$\sigma_{u_3}^2 = \frac{k_2(X_8^2 + Y_8^2)}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega\right\}$$

It is difficult to see the analysis validation for the population of the fluctuation intensity. Nevertheless, can evaluate the mathematical expression for the various

VOL. 17, NO. 10, MAY 2022 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

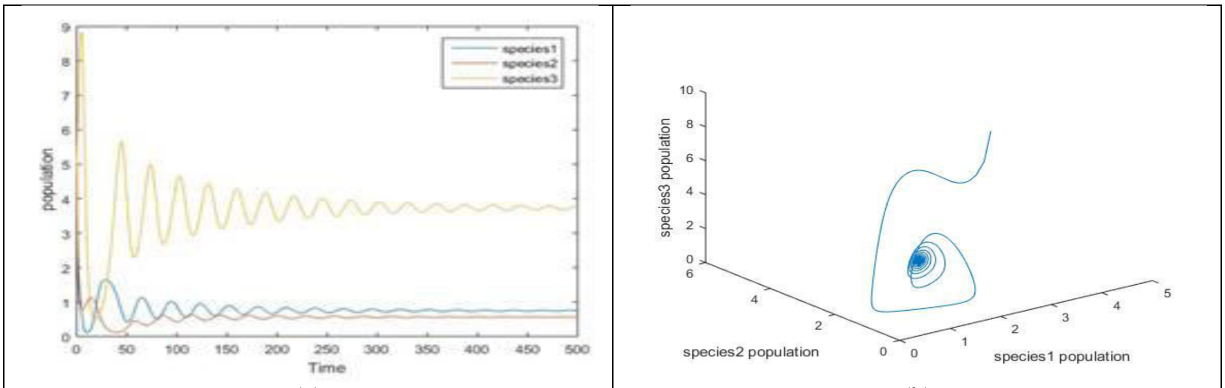
parameters of the function. The simulation results are represented in the Figure-9 to Figure-22.

8. NUMERICAL ILLUSTRATIONS

In this portion, describes the dynamic performance for the proposed frame work (2.1) through the mathematical expression by using the software of MATLAB. Owing to the lack of taking the real time data of all the parameters for the proposed framework the inferable data is taken for the all parameters. The parameter values are given below:

$r = 0.5$, $r_1 = 0.45$, $k = 1.9$, $\alpha = 0.04$, $\beta_1 = 0.09$, s = 0.25, $l=1.4$, $\delta=0.2$, $\beta_2 =$ 0.7850, $\beta_0 =$ 0.49. For these parameter values, it seen that $B_1 =$ 0.119068 > 0, $B_3 =$ 0.005209 > 0, $\Delta = B_1 B_2 - B_3 =$ 0.000852 > 0. As stated in the theorem 11, the positive equilibrium region E_6 becomes asymptotically stabilized in locally state. Figure-1 represents a positive equilibrium region of E_6 on a asymptotically stabilized in locally state.

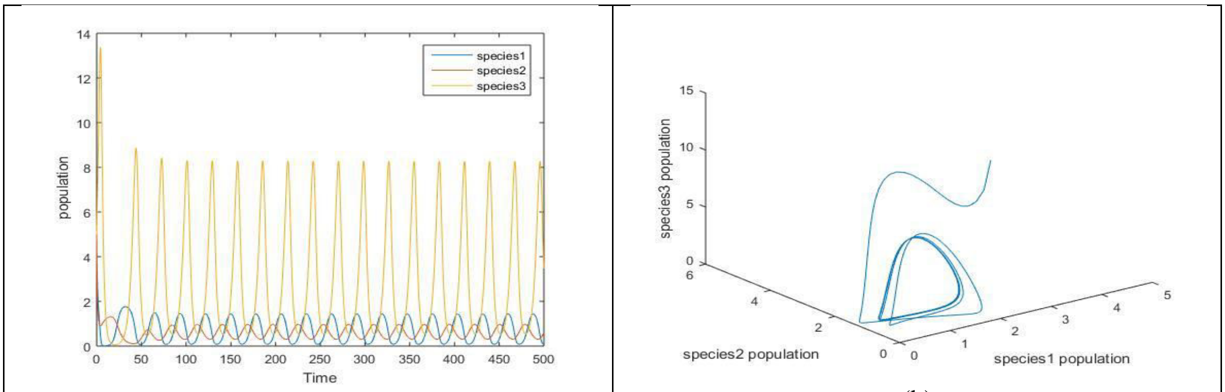
From this figure, it is concluded that the population of all species co-existing concurrently.



(a) (b)

Figure-1. (a) Positive equilibrium region E_6 Stabilitygraph, (b) Two Preys-Predator stable limit cycle.

According to the theorem-6.1, we can define the condemnatory value of β_2. The condemnatory value of β_2^* is 0.9421. The system is unstable for $\beta_2 > \beta_2^*$ around the positive equilibrium point E_6, taking $\beta_2 = 0.95, 1.5, 2.5$, the solution of the system (2.1) has been shown in Figures 4, 6, 8, which indicates that at the point of the positive equilibrium state E_6 the system becomes unstable.



(a) (b)

Figure-2. Bifurcation diagram for $\beta_2 = 0.95$.

VOL. 17, NO. 10, MAY 2022 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

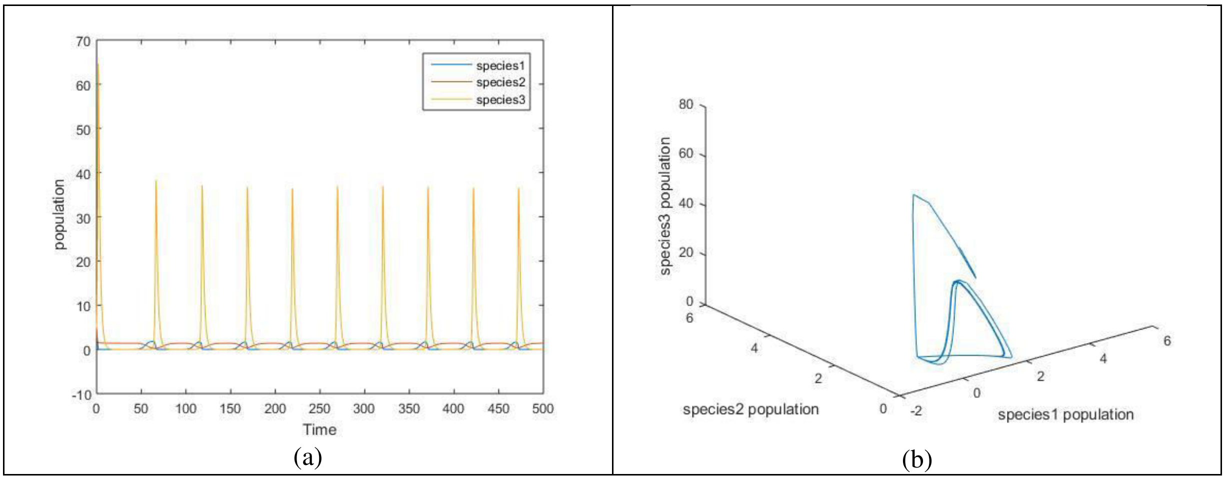


Figure-3. Bifurcation diagram for $\beta_2 = 1.5$.

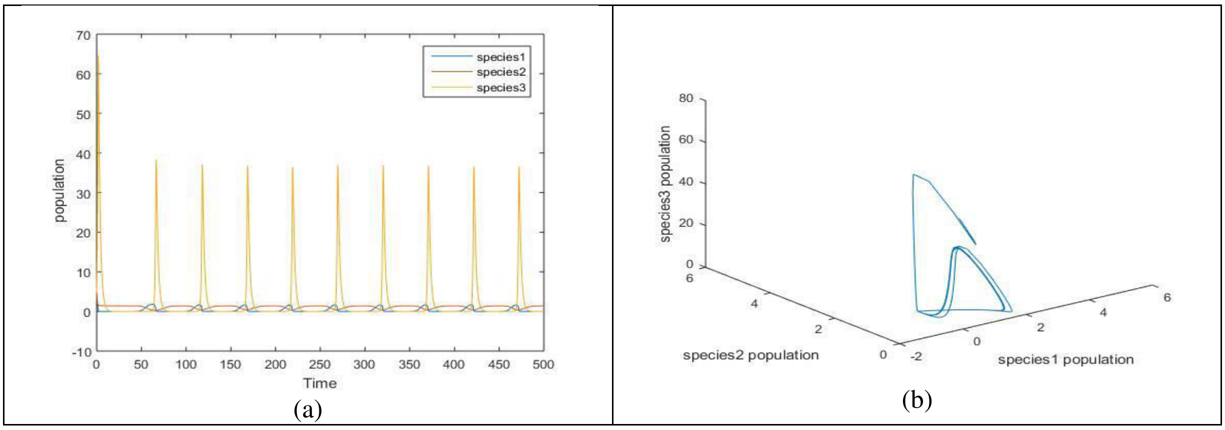


Figure-4. Bifurcation diagram for $\beta_2 = 2.5$.

Case: (a) Consider the parameter values: r = 16.2, r_1 = 15.2, k =4.7, α =1.05 , β_1 = 20, s = 40.64, l =12, δ =10.755 , β_2 = 50.15, β_0 = 3.99, k_1 = 0.1, k_2 = 0.3, k_3 = 0.2, $\xi_1(t)$ = 0.3, $\xi_2(t) = 0.3$, $\xi_3(t) = 0.1$, the stochastic diagram is

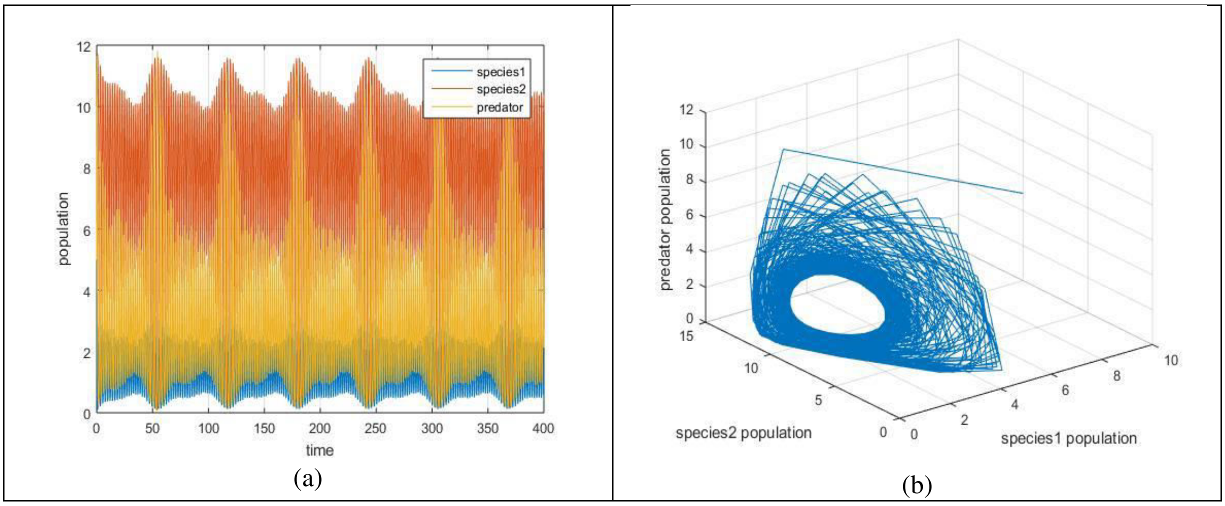


Figure-5. The fluctuating population growth with high intensity on the irregular surroundings.

VOL. 17, NO. 10, MAY 2022 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

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Case: (b) the parameter values are considered as: r =16.2, r_1 = 15.2, k =4.7, α =1.05 , β_1 = 35, s= 40.64, l =12, δ =10.755 , β_2 = 27.5, β_0 = 3.99, k_1 = 0.687, k_2 = 0.743, k_3 = 0.896, $\xi_1(t)$ =0.643, $\xi_2(t)$ = 0.653, $\xi_3(t)$ = 0.757, the relevant stochastic diagram is represented below:

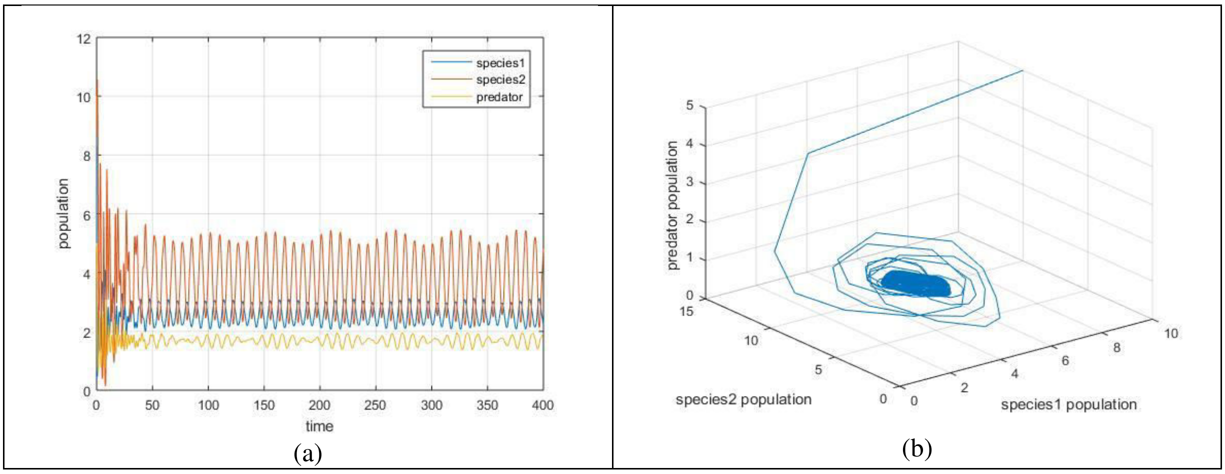

Figure-6. The fluctuating population growth with low intensity on the irregular surroundings.

Case: (c) the parameter values are considered as r = 16.2, r_1 = 7.25, k =4.7, α =1.05 , β_1 = 20, s= 40.64, l =12, δ =10.755, β_2 = 27.5, β_0 = 3.99, k_1 = 5, k_2 = 5, k_3 = 5, $\xi_1(t)$ =5, $\xi_2(t)$ = 3, $\xi_3(t)$ =0.956, the relevant stochastic diagram is represented below:

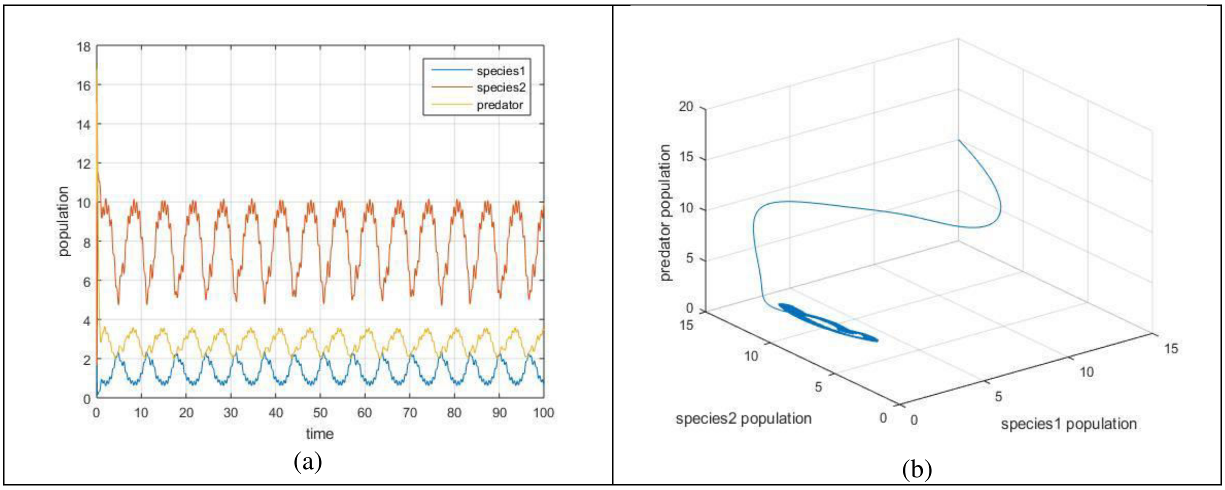

Figure-7. The fluctuating population growth with low intensity on the irregular surroundings.

Case: (d) the parameter values are considered as r = 16.2, r_1 = 16.2, k =4.7, α =1.05 , β_1 = 20, s= 40.64, l =12, δ =10.755, β_2 = 27.5, β_0 = 3.99, k_1 = 6, k_2 = 5, k_3 = 2, $\xi_1(t)$ =3, $\xi_2(t)$ = 5, $\xi_3(t)$ =0.2, the relevant stochastic diagram is represented below:

VOL. 17, NO. 10, MAY 2022 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

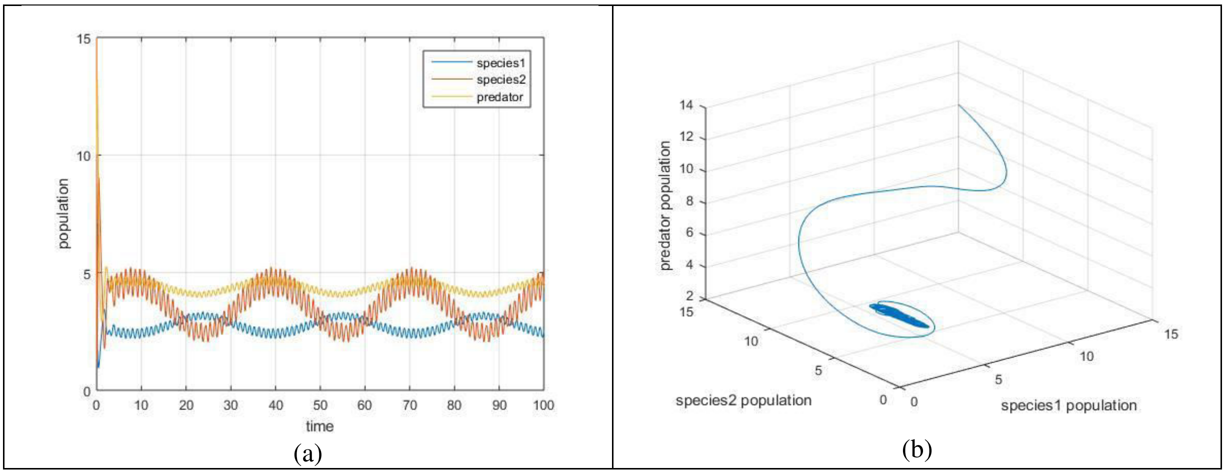

Figure-8. The fluctuating population growth with low intensity on the irregular surroundings.

Case: (e) Here, manystochastic graphs are given with the fluctuating population growth under the low intensity condition for irregular surroundings.

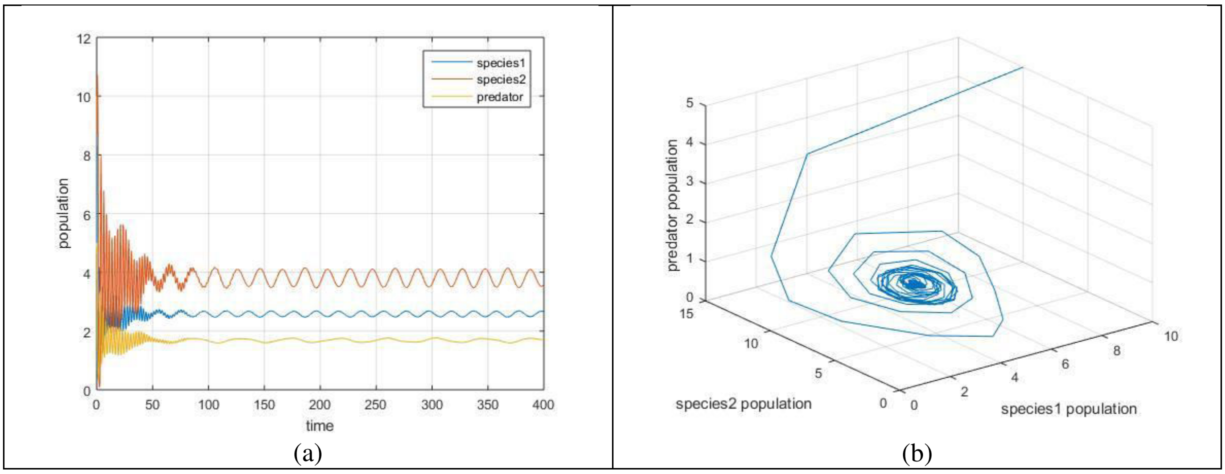

Figure-9. The fluctuating population growth with low intensity on the irregular surroundings.

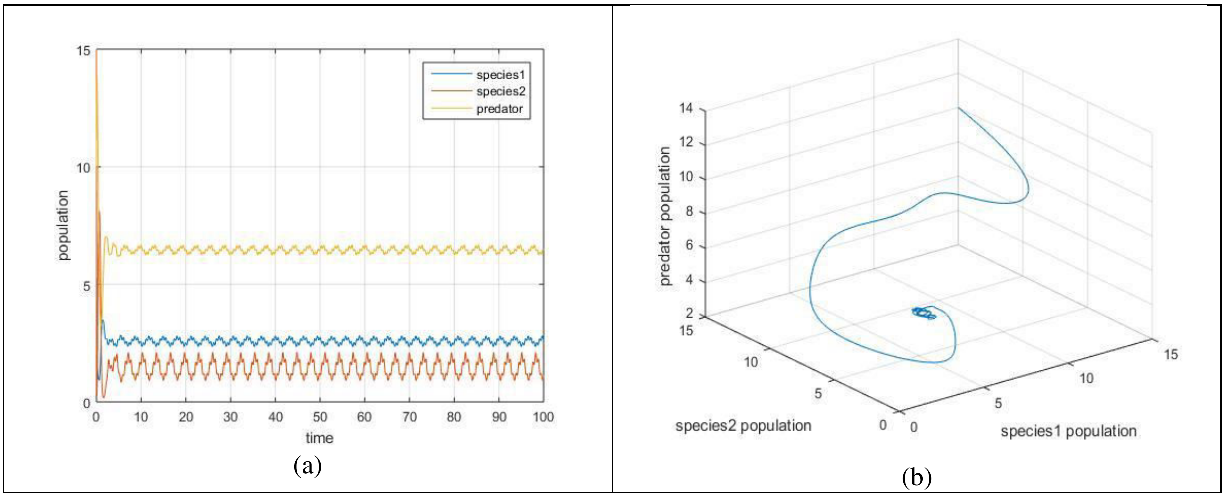

Figure-10. The fluctuating population growth with low intensity on the irregular surroundings.

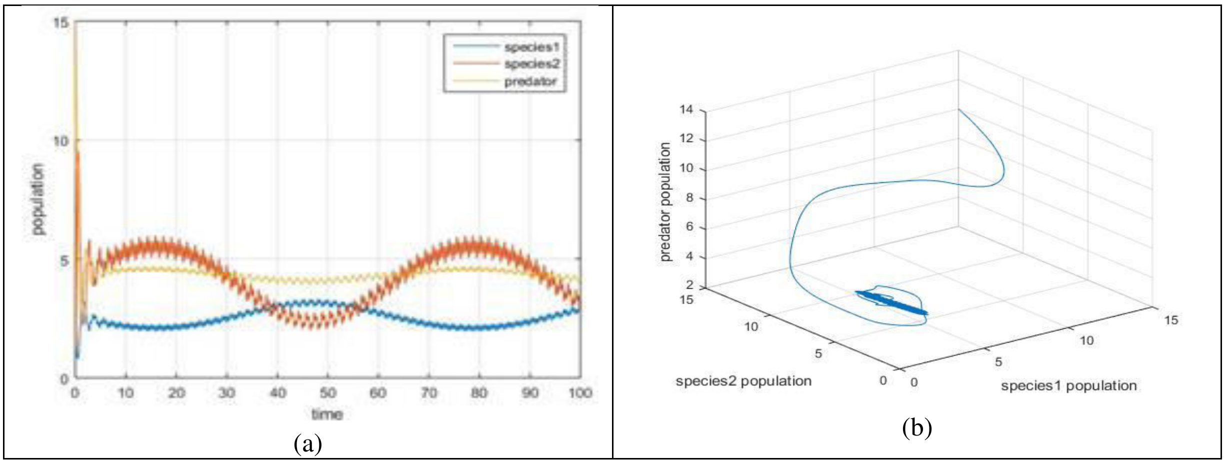

Figure-11. The fluctuating population growth with low intensity on the irregular surroundings.

9. CONCLUSIONS

In this study represented a food chain framework for two preys - hunter population in the ecosystem has been considered. The finite quality of the results and the representing the existence of the points at the equilibrium state are established in this proposed framework. The system performances are analysed in global and local region even at the equilibrium region which is represented in expression (2.1). Hopf bifurcation around the positive equilibrium region has been established. By using the gaussian white noise approach, we developed the proposed stochastic type of framework for the considering the impact of the fluctuating surroundings.

We established the behavior of the system with effect of stochastic perturbations.

By this stochastic process we observed that because of involvement of the stochastic perturbations can form the substantial changes in the intensity of our system framework by the changes in the diplomatic parameters may create a high fluctuation intensity in the surroundings which leads to chaotic behaviour.

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