VOL. 20, NO. 20, OCTOBER 2025 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

INTRODUCING GEOMETRIC DEGREE OF FREEDOM IN STRUCTURAL STABILITY

Nazzal S. Armouti
Department of Civil Engineering, University of Jordan, Amman, Jordan
E-Mail: armouti@ju.edu.jo

ABSTRACT

A geometric degree of freedom, GDOF, is introduced as a means of reducing the size of the problem of structural stability of large multiple DOF frames. GDOFs will be defined as DOFs that are associated with nodes that are associated with members that carry axial loads. It has been shown that the structural stability of frames with a large number of DOF can be reduced to a much smaller size and can be expressed in terms of these declared GDOFs only. These GDOFs may be viewed as analogous to the dynamic DOFs in structural dynamics and carry all the benefits that are associated with them. It has been proven that the behavior of the structure in structural stability using only GDOFs is identical to the behavior of the structural stability using the entire DOFs of the original structure. It has also been shown that the effective basic buckling loads will be equal to the number of GDOFs. The buckling modes higher than the effective basic modes are shown to be too large in order of magnitude to take place, and hence, it would be a waste of resources to obtain them.

Keywords: geometric degrees of freedom, finite element method, Structural stability of frames.

Manuscript Received 12 September 2025; Revised 7 January 2026; Published 20 January 2026

INTRODUCTION

Structural stability of frames is an essential part of frame design in structures. In the past, classical methods of solution were available to treat such problems (Timoshenko and Gere, 1961), (Galambus, 1968). Even though classical methods are rigorous from a theoretical point of view, they are impractical and unmanageable for multistory and multiple bay frames.

Structural stability of frames is practically treated with the geometric stiffness matrix method, which is also considered an approximate method. The geometric stiffness method can also be derived by finite element formulation and may be considered a Finite Element Method, FEM, for stability of frames (Huebner, 1981). As an FEM, the geometric stiffness method must follow FEM rules of size, sensitivity, accuracy, and efficiency. Large frames usually result in a huge number of degrees of freedom (DOFs, and consequently result in large matrices for eigenvalue solution.

In general, the eigenvalue problem of large matrices is regarded as a difficult problem and poses challenges for efficient solutions. (Morgan, 1991), and (Dax, 2021) discuss the difficulty of solving the solution of interior eigenvalue problem for large matrices and propose schemes to improve the approximation and efficiency of this problem. (Michail, 2018) presents the treatment of the eigenvalue of large matrices under random perturbation.

In practice, popular steel codes such as the American Institute of Steel Construction, AISC (AISC, 2023), demand the inclusion of second-order effects in steel frame design. AISC permits the approximate evaluation of the buckling of each column as a separate element from the entire structure, which indicates the need for localized buckling analysis of columns in frames. As will be seen later, the concept of GDOF will be perfectly suited for such analysis.

Consequently, it would be beneficial to reduce the number of degrees of freedom in order to reduce the size of the matrices for simplification of the solution and for computer efficiency. Such a reduction may be utilized in structural stability by introducing the concept of geometric degree of freedom.

Geometric degrees of freedom will be defined in this paper as joints that carry axial loads, whereas the rest of the joints will be defined as usual as static degrees of freedom. It will be proven that it is possible to condense all structure DOFs into several geometric DOFs, which offer a reduced size and more efficient global stiffness and geometric stiffness matrices for the structure. Such concepts and techniques are used in structural dynamics analysis to reduce the size of the relevant matrices using the well-known and well-defined dynamic degrees of freedom.

The full geometric stiffness matrix is often not required, as many diagonals appear with zero values. Therefore, it can be shown that these zero diagonals do not affect the stability analysis and may be condensed and incorporated in the active diagonal elements of the matrix. Consequently, the DOFs that are associated with these active diagonal elements will be introduced as geometric degrees of freedom.

FINITE ELEMENT FORMULATION

The Finite Element Method, FEM, is an approximate solution and is widely used in all fields of science. Application of FEM in the field of stability is well documented. In the case of beam elements, which lead to the application in frames, the solution is given in the form of a stiffness matrix and geometric stiffness

ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

matrix (Chen and Lui, 1987). Considering the sign convention of the generalized forces and displacements shown in Figure-1, the element stiffness matrix and geometric stiffness matrix are given as follows:

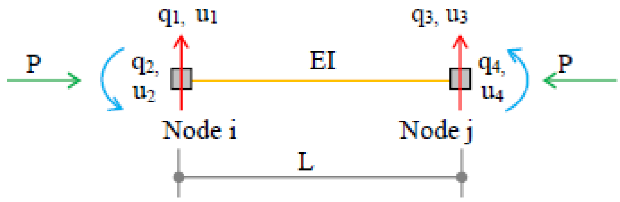


Figure-1. Sign convention of local forces, q, and displacements, u.

$$[k_m] = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \frac{EI}{L^3} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ \hline \end{array}\left[\begin{array}{cccc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right],$$

$$[k_{Gm}] = \begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \frac{P}{30L} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array}\left[\begin{array}{cccc} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{array}\right]$$

where

q_i = generalized element nodal forces which means forces or moments
u_i = generalized element nodal displacements which means displacements or rotations
E = Young modulus
I = moment of inertia of the section about the bending axis
L = length of element
P = externally applied axial load

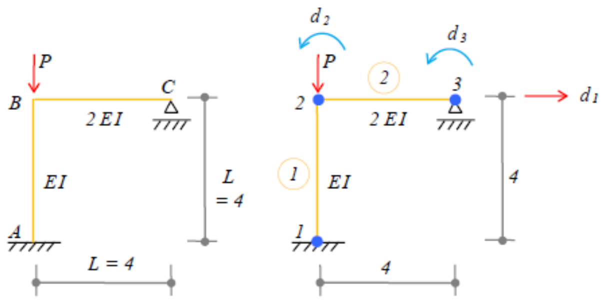
As is well established, assembly of the global stiffness and geometric stiffness matrices, followed by the standard solution of the eigenvalue problem, yields the buckling loads and their buckling shapes of the system.

Consider the axially rigid frame shown in Figure-2(a) which has a fixed support at, A, and a roller at, C, therefore, the frame has 3DOF, one horizontal translation of member, B-C, namely, d_1, and two rotations at, B and, C, namely, d_2 and d_3, as shown in Figure-2(b).

The global stiffness and geometric stiffness matrices may be constructed by inspection or, by systematic assembly. If the assembly process is used, the global stiffness matrix [K] can be constructed in conjunction with the element matrices, $[k_m]$, and the kinematic matrices, [T], presented in (Chen and Lui, 1987), which take the following form

$$[K] = \Sigma\ [T]^T [k_m] [T]$$

The frame may be discretized into two elements and three nodes as shown in Figure-3. If the member incidences are taken as sequentially numbered in the figure, the formulation of the problem proceeds as follows:



(a) Frame detail (b) FEM model

Figure-2. Frame layout EI = 1'000 kN.m^2.

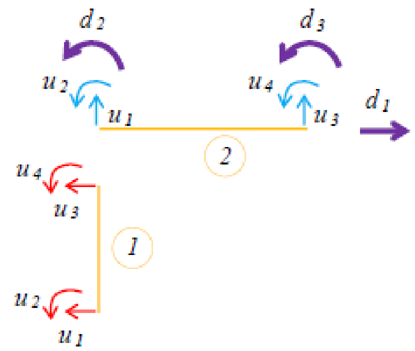


Figure-3. Frame discretization into two elements.

$$[T_1]^T = \begin{array}{c} \\ d_1 \\ d_2 \\ d_3 \end{array} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array}\left[\begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad [T_2]^T = \begin{array}{c} \\ d_1 \\ d_2 \\ d_3 \end{array} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array}\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Where

u_i = element nodal displacement as defined in Figure-3
d_i = global nodal displacement as defined in Figure-3
T_i = kinematic matrix that relates element nodal displacements to global nodal displacement

With the element matrices $[k_m]$ and $[k_{Gm}]$ are calculated as follows:

$$[k_{m1}] = \frac{EI}{L^3}\left[\begin{array}{cccc} 12 & -6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right] = \left[\begin{array}{cccc} 187.5 & 375 & -187.5 & 375 \\ 375 & 1'000 & -375 & 500 \\ -187.5 & -375 & 187.5 & -375 \\ 375 & 500 & -375 & 1'000 \end{array}\right]$$

VOL. 20, NO. 20, OCTOBER 2025 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

$$[k_{m2}] = \frac{2EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} = \begin{bmatrix} 375 & 750 & -375 & 750 \\ 750 & 2'000 & -750 & 1'000 \\ -375 & -750 & 375 & -750 \\ 750 & 1'000 & -750 & 2'000 \end{bmatrix}$$

$$[k_{Gm1}] = \frac{P}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} = P\begin{bmatrix} 0.3 & 0.1 & -0.3 & 0.1 \\ 0.1 & 0.533 & -0.1 & -0.133 \\ -0.3 & -0.1 & 0.3 & -0.1 \\ 0.1 & -0.133 & -0.1 & 0.533 \end{bmatrix}$$

$[k_{Gm2}] = [0]_{4x4}$, no axial load in member 2

Consequently, the global matrices are obtained as follows:

$$[K] = T_1{}^T k_{m1} T_1 + T_2{}^T k_{m2} T_2 = \left[\begin{array}{cc:c} 187.5 & 375 & 0 \\ 375 & 3,000 & 1,000 \\ \hdashline 0 & 1,000 & 2,000 \end{array}\right]$$

$$[K_G] = T_1{}^T k_{Gm1} T_1 = P\left[\begin{array}{cc:c} 0.3 & 0.1 & 0 \\ 0.1 & 0.533 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right]$$

Note that the third row and third column and their diagonal in the geometric stiffness matrix have zero values, which by inspection indicates that this DOF, d_3, which is not associated with any axially loaded member, does not participate in the geometric stiffness matrix, and hence, is not considered a geometric degree of freedom. On the other hand, the other two DOFs associated with the axially loaded member participate in the geometric stiffness matrix, and hence, will be declared geometric degrees of freedom GDOF.

The characteristic equation of the original system is given as follows:

$$\{[K] - [K_G]\}\{D\} = \{0\}$$

$$\left\{\left[\begin{array}{cc:c} 187.5 & 375 & 0 \\ 375 & 3,000 & 1,000 \\ \hdashline 0 & 1,000 & 2,000 \end{array}\right] - P\left[\begin{array}{cc:c} 0.3 & 0.1 & 0 \\ 0.1 & 0.533 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right]\right\}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left[\begin{array}{c:c:c} 187.5-0.3P & 375-0.1P & 0-0P \\ 375-0.1P & 3,000-0.533P & 1,000-0P \\ 0-0P & 1,000-0P & 2,000-0P \end{array}\right]\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This is an eigenvalue problem which yields the following results for a non-trivial solution for det. [ ] = 0, as follows:

$$\left|\left[\begin{array}{c:c:c} 187.5-0.3P & 375-0.1P & 0 \\ 375-0.1P & 3,000-0.533P & 1,000 \\ 0 & 1,000 & 2,000 \end{array}\right]\right| = 0$$

or,

$$(187.5 - 0.3\,P)\,\{(3,000 - 0.533\,P)\,(2,000) - 1,000^2\}$$
$$-\ (375 - 0.1\,P)\,\{(375 - 0.1\,P)\,(2,000)\} + 0 = 0$$

Simplifying the expression above leads to

$$P^2 - 5,167\,P + 2,187,500 = 0$$

Note that the resulting equation above is a quadratic equation that yields two eigenvalues, which lead to two buckling loads and two buckling modes only, even though the total number of DOFs is three. The solution results are obtained as follows:

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 465 \\ 4,701 \end{Bmatrix}$$

Back substitution in the characteristic equation, yields two buckling modes as follows:

$$[\,\phi\,] = \left[\phi_1 \,\vdots\, \phi_2\right] = \left[\begin{array}{c:c} 1 & 1 \\ -0.146 & -12.85 \\ 0.073 & +6.427 \end{array}\right]$$

It should be pointed out at this stage that the number of buckling loads are two, which are equal to the declared number of GDOFs, and not equal to the total number of DOFs. This observation implies that the number of buckling loads is function of the number of declared GDOFs only.

Therefore, for large structures, the formulation and analysis of the bucking problem can be reduced from the size of the total number of degrees of freedom to the size of the number of geometric degrees of freedom which is, in fact, advantageous in reducing the solution time in the computers on one hand and eliminates the problem of singularities resulting from having zero diagonals in the geometric stiffness matrix on the other.

The reduction of the size of the buckling problem from the total number of DOFs to the number of geometric DOFs can be accomplished by two methods: (1) the Kinematic condensation method, and (2) the Flexibility method, as will be presented in the following sections.

- **Kinematic Condensation Method**

Inspection of the global matrices, [K] and $[K_G]$, obtained in the previous section, reveals that the geometric stiffness matrix contains non-zero elements related to the geometric DOFs only, while all other elements are zeros, therefore, the rows and columns of zero values can be eliminated as follows:

VOL. 20, NO. 20, OCTOBER 2025 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

$$[K_G]_{orignal} = P\left[\begin{array}{cc|c} 0.3 & 0.1 & 0 \\ 0.1 & 0.533 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow [K_G]_{reduced} = P\begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.533 \end{bmatrix}$$

However, the stiffness matrix, [K], cannot be reduced by simply eliminating the third row and third column because the stiffness of the system as a whole is affected by these elements. Therefore, their effect may be incorporated indirectly in the stiffness of the system by the construction of a new matrix with a size equals to GDOFs, and at the same time represents the stiffness of the system. In fact, such a matrix does exist, and is known as a kinematically condensed matrix, which can be obtained as follows.

Recall that the characteristic equation was derived earlier as follows:

$$\left\{\left[\begin{array}{cc|c} 187.5 & 375 & 0 \\ 375 & 3{,}000 & 1{,}000 \\ \hline 0 & 1{,}000 & 2{,}000 \end{array}\right] - P\left[\begin{array}{cc|c} 0.3 & 0.1 & 0 \\ 0.1 & 0.533 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]\right\}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

If the matrices are partitioned as shown above and written symbolically as follows:

$$\left\{\left[\begin{array}{c|c} [K_{11}] & [K_{12}] \\ \hline [K_{21}] & [K_{22}] \end{array}\right] - P\left[\begin{array}{c|c} [K_{G11}] & [0] \\ \hline [0] & [0] \end{array}\right]\right\}\begin{Bmatrix} \{d_1\} \\ \{d_o\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix}$$

And then, by multiply the bottom row through yields the following:

$$\{\{K_{21}\} - P\,[0]\}\{d_1\} + \{[K_{22}] - P\,[0]\}\,\{d_o\} = \{0\}$$

or, $[K_{21}]\,\{d_1\} + [K_{22}]\,\{d_o\} = \{0\}$

or, $\{d_o\} = -\,[K_{22}]^{-1}\,[K_{21}]\,\{d_1\}$ … (1)

And by multiply the upper row through yields the following:

$$\{[K_{11}] - P\,[K_{G11}]\}\{d_1\} + \{[K_{12}] - P\,[0]\}\,\{d_o\} = \{0\} \quad \ldots (2)$$

Substitution of, $\{d_o\}$, from (1) into (2), yields

$$\{[K_{11}] - P\,[K_{G11}]\}\{d_1\} + [K_{12}]\,\{-[K_{22}]^{-1}[K_{21}]\,\{d_1\}\} = \{0\}$$

or, $\{[K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}] - P\,[K_{G11}]\}\{d_1\} = \{0\}$

The above equation indicates that the condensed stiffness matrix can now be given as:

$$[K]_{condensed} = [[K_{11}] - [K_{12}]\,[K_{22}]^{-1}[K_{21}]]_{original}$$

Going back to our example, the condensed stiffness matrix can be obtained as follows:

$$[K_{11}] = \begin{bmatrix} 187.5 & 375 \\ 375 & 3{,}000 \end{bmatrix},\ [K_{12}] = \begin{bmatrix} 0 \\ 1{,}000 \end{bmatrix},\ [K_{21}] = \begin{bmatrix} 0 & 1{,}000 \end{bmatrix}$$

$$[K_{22}] = [2{,}000] \rightarrow [K_{22}]^{-1} = \left[\frac{1}{2'000}\right]$$

hence,

$$[K]_{condensed} = [[K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}]]_{original}$$

$$\text{or, } [K]_{condensed} = \begin{bmatrix} 187.5 & 375 \\ 375 & 3{,}000 \end{bmatrix} - \begin{bmatrix} 0 \\ 1{,}000 \end{bmatrix}\left[\frac{1}{2'000}\right]\begin{bmatrix} 0 & 1{,}000 \end{bmatrix}$$

$$\text{or, } [K]_{condensed} = \begin{bmatrix} 187.5 & 375 \\ 375 & 2{,}500 \end{bmatrix}$$

The characteristic equation is accordingly given as follows:

$$\{[K] - [K_G]\}\ \{D\} = \{0\}$$

$$\left\{\begin{bmatrix} 187.5 & 375 \\ 375 & 2{,}500 \end{bmatrix} - P\begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.533 \end{bmatrix}\right\}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left[\begin{array}{c|c} 187.5 - 0.3\,P & 375 - 0.1\,P \\ 375 - 0.1\,P & 2{,}500 - 0.533\,P \end{array}\right]\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This is an eigenvalue problem which yields the following results for non-trivial solution for det. [ ] = 0, as follows:

$$\left|\left[\begin{array}{c|c} 187.5 - 0.3\,P & 375 - 0.1\,P \\ 375 - 0.1\,P & 2{,}500 - 0.533\,P \end{array}\right]\right| = 0$$

or,

$$(187.5 - 0.3\,P)(2{,}500 - 0.533\,P) - (375 - 0.1\,P)^2 = 0$$

Simplifying the expression above leads to

$$P^2 - 5{,}167\,P + 2{,}187{,}500 = 0$$

Note that the resulting equation above is exactly the equation obtained earlier for total DOFs, hence, the solution results will be identical and given as follows:

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 465 \\ 4{,}701 \end{Bmatrix}$$

Back substitution in the characteristic equation, yields the two modes as follows:

$$[\phi] = [\phi_1 \mid \phi_2] = \left[\begin{array}{c|c} 1 & 1 \\ -0.146 & -12.85 \end{array}\right]$$

Note that the mode shapes are obtained in terms of geometric DOFs, which are d_1 and d_2.

ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

Note also that the reduced geometric stiffness matrix can be obtained directly in terms of the geometric DOFs, d_1 and d_2, which involves member #1 only, as presented in Figure-4 which is obtained as follows:

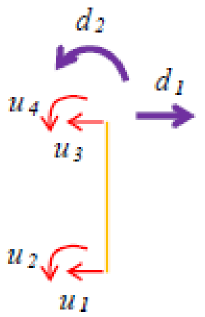


Figure-4. Relationship between local and global DOFs (Geometric DOFs only).

$$[T_1]^T = \begin{array}{c} \\ d_1 \\ d_2 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$[K_{Gm1}] = P \begin{bmatrix} 0.3 & 0.1 & -0.3 & 0.1 \\ 0.1 & 0.533 & -0.1 & -0.133 \\ -0.3 & -0.1 & 0.3 & -0.1 \\ 0.1 & -0.133 & -0.1 & 0.533 \end{bmatrix}$$

hence,

$$[K_G] = T_1{}^T k_{Gm1} T_1 = P \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.533 \end{bmatrix}$$

The same results as before!

- **Flexibility Method**

Since the condensation method produces a relationship between the global nodal forces, $\{Q\}$, and the global displacements, $\{D\}$, of the stiffness matrix in terms of the geometric DOFs (of size, m x m), this relationship may be expressed as follows

$$\{Q\}_{m x 1} = [K]_{m x m} \{D\}_{m x 1} \quad \ldots (1)$$

By premultiplying the equation above by, $[K]^{-1}$, yields the following:

$[K]^{-1} \{Q\} = [K]^{-1} [K] \{D\}$
or, $\{D\} = [K]^{-1} \{Q\} = [F]_{m x m} \{Q\}$

hence, $\{Q\}_{m x 1} = [F]^{-1}{}_{m x m} \{D\}_{m x 1} \quad \ldots (2)$

Comparing Eqs. (1) and (2) above, it can be deduced that the following matrices are equal

$$[K]_{m x m} = [F]^{-1}{}_{m x m}$$

Where, $[F]$, is a flexibility matrix that relates the displacements of the nodal GDOFs to the forces of the GDOFs, which, by definition, can be constructed by applying a unit force at each GDOF and then calculating the resulting displacements. This procedure is a big advantage in large structures, as pointed out earlier.

To illustrate this concept, consider the frame in this example, which can be analyzed using any convenient software. Such an analysis would produce the results shown in Figure-5.

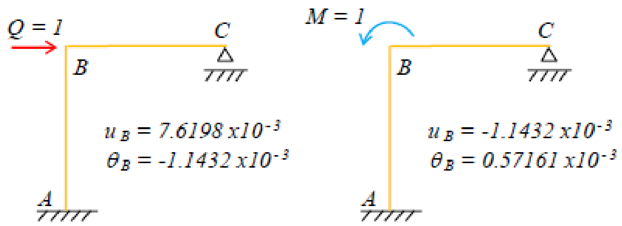


Figure-5. Flexibility matrix elements due to unit force and unit moment at Joint 1 only.

a) By applying a unit force in direction of d_1, the displacement and rotation at, B, are obtained as shown in the figure.

b) By applying a unit moment in direction of, d_2, the displacement and rotation at, B, are obtained as shown in the figure.

Inspection of the above figure reveals that the flexibility matrix, $[F]$, in terms of, d_1 and d_2, is given as follows

$$[F] = 10^{-3} \begin{bmatrix} 7.6190 & -1.1429 \\ -1.1429 & 0.5714 \end{bmatrix}$$

Taking the inverse of, $[F]$, the stiffness matrix, $[K]$, is obtained as follows:

$$[K] = [F]^{-1} = \begin{bmatrix} 187.5 & 375 \\ 375 & 2'500 \end{bmatrix}$$

which is the same answer obtained by the kinematic condensation method.

It should be kept in mind that the geometric stiffness matrix can still be constructed directly using the geometric DOFs as given under the kinematic condensation method.

APPLICATION IN LARGE FRAMES AND DISCUSSION

As an example of the size reduction and efficiency benefits that can be gained from consideration of geometric degrees of freedom, consider the frame shown in Figure-6. The frame has three bays and two

VOL. 20, NO. 20, OCTOBER 2025 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

levels and is supported with fixations at A, B, C, and D as shown. The frame is subjected to a vertical load, P, along the column line, A. For analysis purposes, all members are assigned to the same flexure stiffness with EI = 10,000 kN.m^2, and area to moment of inertia ratio, A/I =12. It is intended to find the critical load of this frame under this loading using the FEM with the complete DOFs and then using only the geometric DOFs in the analysis, and then make a comparison between the results.

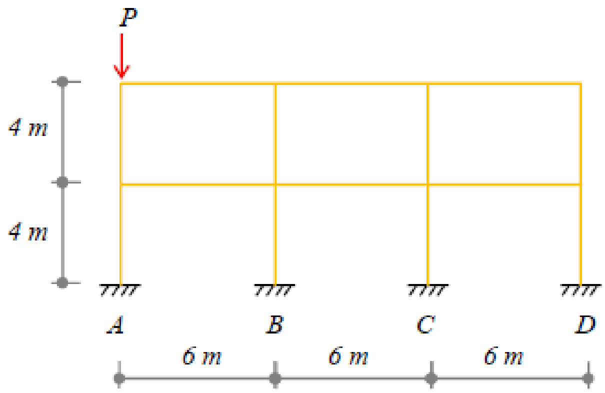


Figure-6. Frame arrangement EI = 10,000 kN.m^2, A/I = 12 (all members).

Analysis Using the Complete Model with 24 DOFs

Figure-7 presents a numbering scheme for the analysis. Nodes are designated with numbers without circles from 1 to 12, while members are designated with numbers in circles from 1 to 14. Since each free node has 3 DOFs, horizontal, vertical, and rotational, and there are 8 free nodes, the frame will have a total of 24 DOFs. Analyzing the frame with these 24DOFs results in eigenvalue matrices of size 24x24. Such an analysis can be performed using any available software program that can perform such an analysis. After running this model, the resulting 24 eigenvalues (buckling loads) and their associated eigenvectors (buckling shapes) are obtained as shown in Table-1. For comparison purposes, the geometric DOFs, which are the horizontal displacement and the rotation of nodes 2 and 3, are highlighted in the table with underline and bold fonts.

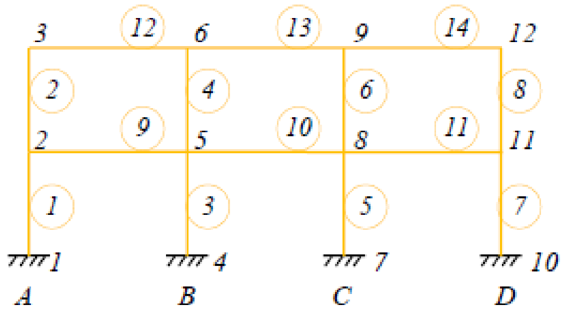


Figure-7. Frame model numbering scheme: Numbers without circles = node, Numbers with circles = member.

VOL. 20, NO. 20, OCTOBER 2025 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

Table-1. Computer output of frame eigenvalue solution using 24 DOFs (Geometric DOFs are underlined with bold font).

	Buckling load 1 +.10773e+05	**Buckling load 2 +.14291e+05**	**Buckling load 3 +.18874e+05**	**Buckling load 4 +.41416e+05**	**Buckling load 5 +.11562e+20**
Node	**Mode 1**	**Mode 2**	**Mode 3**	**Mode 4**	**Mode 5**
1	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
2	**+.16459e+00**	**+.10000e+01**	**+.10000e+01**	**-.58127e-01**	**+.68654e-16**
	+.19900e-01	+.55109e-02	+.74303e-02	-.73047e-01	-.61662e-02
	-.10966e+00	**-.35961e+00**	**+.43971e+00**	**+.58693e+00**	**-.35309e-16**
3	**+.10000e+01**	**+.78087e+00**	**+.90768e+00**	**-.18782e+00**	**+.25600e-15**
	+.29992e-01	-.11187e-01	+.30783e-01	-.11742e+00	-.16496e-01
	-.12414e+00	**+.35157e+00**	**-.53967e+00**	**+.10000e+01**	**+.13777e-16**
4	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
5	+.22669e+00	+.81216e+00	+.84341e+00	-.76297e-01	+.50681e-01
	-.42943e-02	+.27330e-02	+.16858e-02	+.75461e-01	+.67410e-02
	-.74928e-01	-.41066e-01	-.15501e+00	-.21259e-01	-.41762e-01
6	+.91208e+00	+.80637e+00	+.91585e+00	-.15048e+00	-.74065e-01
	-.70297e-02	+.21056e-01	-.25247e-01	+.12315e+00	+.28369e-01
	-.69858e-01	-.40262e-01	+.10064e+00	-.12114e+00	+.12923e+00
7	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
8	+.25889e+00	+.70854e+00	+.74023e+00	-.71952e-01	+.82908e-01
	+.11452e-02	+.17018e-02	+.29990e-02	-.50727e-02	-.43067e-01
	-.71727e-01	-.78405e-01	-.73077e-01	+.10640e-01	+.14378e+00
9	+.86126e+00	+.81607e+00	+.92062e+00	-.14679e+00	-.13698e+00
	+.16931e-02	+.32370e-04	+.82875e-02	-.96108e-02	-.11123e+00
	-.59659e-01	+.39910e-02	-.25021e-01	+.17206e-01	-.30075e+00
10	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00	+.00000e+00
11	+.27008e+00	+.66053e+00	+.69422e+00	-.70563e-01	+.31608e+00
	-.16750e-01	-.99457e-02	-.12115e-01	+.26587e-02	+.42492e-01
	-.93458e-01	-.10529e+00	-.11928e+00	+.17787e-01	-.27424e+00
12	+.84228e+00	+.82191e+00	+.92391e+00	-.14490e+00	-.31851e+00
	-.24655e-01	-.99020e-02	-.13824e-01	+.38813e-02	+.99355e-01
	-.91411e-01	-.65163e-02	-.13103e-01	+.92984e-02	+.10000e+01

Inspection of Table-1 indicates that the first four modes are of significant values. The first four buckling loads range from 10,773 to 41,416 kN. However, the value of the fifth mode is extremely high that reaching an exponent of, 20, which indicates that the number of effective modes equals the number of geometric DOFs only.

Analysis Using Geometric DOFs Only With 4 GDOFs

Referring to Figures 6 and 7, it can be seen that the only nodes associated with members with axial loads are nodes, 2 and 3, as shown in Figure-7, and hence the GDOFs will be the horizontal displacement and rotation of these two nodes only. Figure-8 shows the resulting four GDOFs, d_1, d_2, d_3, and d_4. Consequently, it will be only required to construct an eigenvalue problem with order of 4, i.e., four buckling loads and associated four buckling shapes.

VOL. 20, NO. 20, OCTOBER 2025 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

www.arpnjournals.com

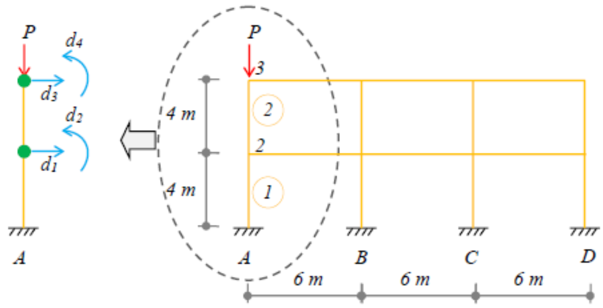


Figure-8. Frame four geometric DOFs d_1, d_2, d_3, d_4.

The condensed stiffness matrix may be obtained by taking the inverse of a condensed flexibility matrix. The flexibility matrix can be constructed by applying a unit force at each GDOF and calculating all displacements and rotations associated with this unit force. For example, the first column of the flexibility matrix can be obtained by applying a unit horizontal force at Joint 2. Figure-9 shows the results of a software solution of the application of a unit force at Joint 2. This procedure is repeated for each GDOF as shown in Figure-10. The results of Figure-9 and Figure-10 are then compiled to yield the 4x4 condensed flexibility matrix in terms of GDOFs as follows:

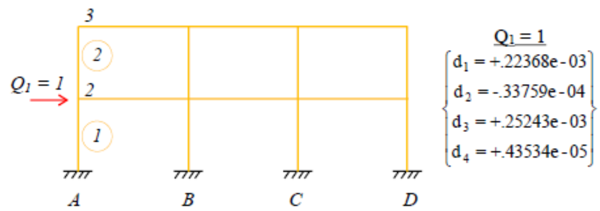


Figure-9. Flexibility matrix elements due to unit force at joint 2.

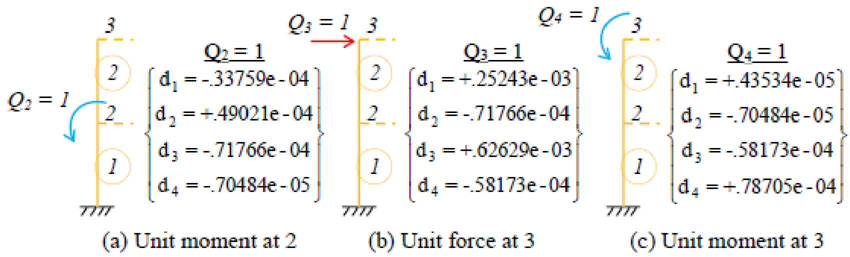


Figure-10. Flexibility matrix elements due to unit forces and moments at Joint 2 and Joint 3.

$$[F]=10^{6}\begin{bmatrix} +223.68 & -33.759 & +252.43 & +4.3534 \\ -33.759 & +49.021 & -71.766 & -7.0484 \\ +252.43 & -71.766 & +626.29 & -58.173 \\ +4.3534 & -7.0484 & -58.173 & +78.705 \end{bmatrix}$$

Taking the inverse of the flexibility matrix above, yields the condensed 4x4 stiffness matrix as follows:

$$[K]=[F]^{-1}=\begin{bmatrix} +8,977.51 & -86.42 & -3,946.1 & -3,420.98 \\ -86.247 & +26,157.3 & +3,489.79 & +4,926.68 \\ -3,946.1 & +3,489.79 & +3,904.46 & +3,416.69 \\ -3,420.98 & +4,926.68 & +3,416.69 & +15,861.5 \end{bmatrix}$$

The geometric stiffness matrix in terms of GDOFs is obtained as presented earlier using the kinematic matrix principle. Figure-11 shows the relationships between the local and global DOFs of the active members associated with the GDOFs. Accordingly, the kinematic matrices for members 1 and 2 are constructed as follows:

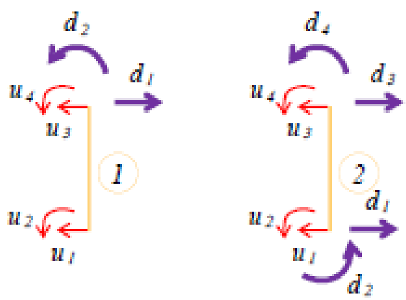


Figure-11. Relationship between local and global DOFs (Kinematic matrices, T_1 and T_2).

VOL. 20, NO. 20, OCTOBER 2025 ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

$$[T_1]^T = \begin{array}{c c} & \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{array}{c} d_1 \\ d_2 \\ d_3 \\ d_4 \end{array} & \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad [T_2]^T = \begin{array}{c c} & \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{array}{c} d_1 \\ d_2 \\ d_3 \\ d_4 \end{array} & \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The element geometric matrices for members 1 and 2 are obtained as follows. Note that, in this case, members 1 and 2 have the same dimensions and properties, and hence they have the same element geometric matrix, i.e.

$$[k_{Gm1}] = [k_{Gm2}] = \frac{P}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

$$= P\begin{bmatrix} 0.3 & 0.1 & -0.3 & 0.1 \\ 0.1 & 0.533 & -0.1 & -0.133 \\ -0.3 & -0.1 & 0.3 & -0.1 \\ 0.1 & -0.133 & -0.1 & 0.533 \end{bmatrix}$$

Using the assembly equation presented earlier

$$[K_G] = \Sigma[T]^T[k_{Gm}]\,[T]$$

$$[K_G] = T_1{}^T k_{Gm1}\, T_1 + T_2{}^T k_{Gm2} T_2 = P\begin{bmatrix} +0.6 & 0 & -0.3 & -0.1 \\ 0 & +1.0667 & +0.1 & -0.133 \\ -0.3 & +0.1 & +0.3 & +0.1 \\ -0.1 & -0.133 & +0.1 & +0.533 \end{bmatrix}$$

The characteristic equation of the original system in its condensed form is now given as follows:

$$\{[K] - [K_G]\}\,\{D\} = \{0\}$$

$$\left\{\begin{bmatrix} +8,977.51 & -86.42 & -3,946.1 & -3,420.98 \\ -86.247 & +26,157.3 & +3,489.79 & +4,926.68 \\ -3,946.1 & +3,489.79 & +3,904.46 & +3,416.69 \\ -3,420.98 & +4,926.68 & +3,416.69 & +15,861.5 \end{bmatrix} - P\begin{bmatrix} +0.6 & 0 & -0.3 & -0.1 \\ 0 & +1.0667 & +0.1 & -0.133 \\ -0.3 & +0.1 & +0.3 & +0.1 \\ -0.1 & -0.133 & +0.1 & +0.533 \end{bmatrix}\right\}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The resulting equation above is a characteristic equation of order 4, which leads to four buckling loads and four associated buckling modes only, even though the total number of DOFs is 24. The solution results are given as follows:

Eigenvalues (Four buckling loads)

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} 10,772 \\ 14,289 \\ 18,874 \\ 41,416 \end{Bmatrix}$$

Eigenvectors (Four buckling shapes)

$$[\,\phi\,] = \left[\phi_1 \,\vdots\, \phi_2 \,\vdots\, \phi_3 \,\vdots\, \phi_4\right] = \begin{bmatrix} 0.16459 & 1 & 1 & -0.05827 \\ -0.10966 & -0.35946 & 0.44007 & 0.58692 \\ 1 & 0.78090 & 0.90765 & -0.18789 \\ -0.12415 & 0.35143 & -0.54000 & 1 \end{bmatrix}$$

In comparison between the analysis with only these four effective basic GDOFs, and the analysis with the entire original 24 DOFs as presented in Table-1, it can be seen that the results are identical. If the reader observes any negligible deviations with some numbers, it is because the analysis with 24 DOFs is conducted with a software program that carries the entire operations internally and hence using the total digit numbers in the operations, whereas, the analysis with GDOFs is conducted with separate software programs to show the details of intermediate analysis steps and hence, the numbers are carried with only significant digits from program to program.

The comparison between the two analysis methods shows identical results in terms of both the lowest four buckling loads and their associated buckling shapes are identical. Furthermore, it is worth noting that the fifth mode, and of course, the modes higher than the fifth, obtained from complete analysis, are way much higher than the first four modes in order of magnitude, indicating that modes higher than the first four are immaterial. It can be further concluded that effective buckling modes are equal and related to the GDOFs only.

CONCLUSIONS

It has been demonstrated that, in matrix analysis of structural stability, the DOFs of the nodes that are associated with axially loaded members are the only nodes needed for the analysis, and hence can be declared geometric degrees of freedom. The rest of the nodes reflect the stiffness of the structure, but not the second order effect in buckling. It has also been shown that the resulting global matrix size can be reduced to the size of the geometric degrees of freedom.

This reduction in the size of the matrices can be accomplished by two procedures, the kinematic condensation method and the flexibility method. While kinematic condensation is suited for systematic computer programming, the flexibility method is beneficial for office use with available simple structural software programs. It has been shown that both methods, kinematic condensation and flexibility, yield reduced matrices sizes

ISSN 1819-6608
ARPN Journal of Engineering and Applied Sciences

that reflect the exact behavior of the structure using the original large matrices.

It has also been shown that the effective number of buckling modes equals the number of geometric DOFs. It has also been shown that, in complete DOFs, the buckling modes higher than those associated with GDOFs are unlikely to take place as their values are huge in an order of magnitude with respect to the basic modes associated with GDOFs.

To better grasp this concept, it will be useful to draw an analogous comparison with the dynamic DOFs in structural dynamic analysis. As it was beneficial to deal with dynamic DOFs in structural dynamics for decades, using geometric DOFs in structural stability carries the same benefits and advantages that dynamic DOFs offer in the analysis.

REFERENCES

AISC. 2023. Specification for Structural Steel Buildings. American Institute of Steel Construction, Chicago, IL 60601.

Chen W. F. and Lui E. M. 1987. Structural Stability, Theory and Implementation. Elsvier, New York.

Dax A. 2021. Computing interior eigenvalues of large, sparse symmetric matrices. Int. J. Appl. Comput. Math. 7(256): 1-15.

Galambus T. V. 1968. Structural Members and Frames. Prentice-Hall, Englewood, Cliffs, New Jersey.

Huebner K. H. and Thornton E. A. 1981. The Finite Element Method for Engineers. 2nd ed., John Wiley & Sons, New York.

Michail A. 2018. Eigenvalues and eigenvectors of large matrices under random perturbations. Functional Analysis, Sorbonne University, Paris. English. NNT: 2018USPCB214.

Morgan B. R. 1991. Computing interior eigenvalues of large matrices. Linear algebra and its applications, SIAM Review. 154-156: 289-309.

Timoshenko S. P. and Gere J. M. 1961. Theory of elastic stability. McGraw-Hill, New York.